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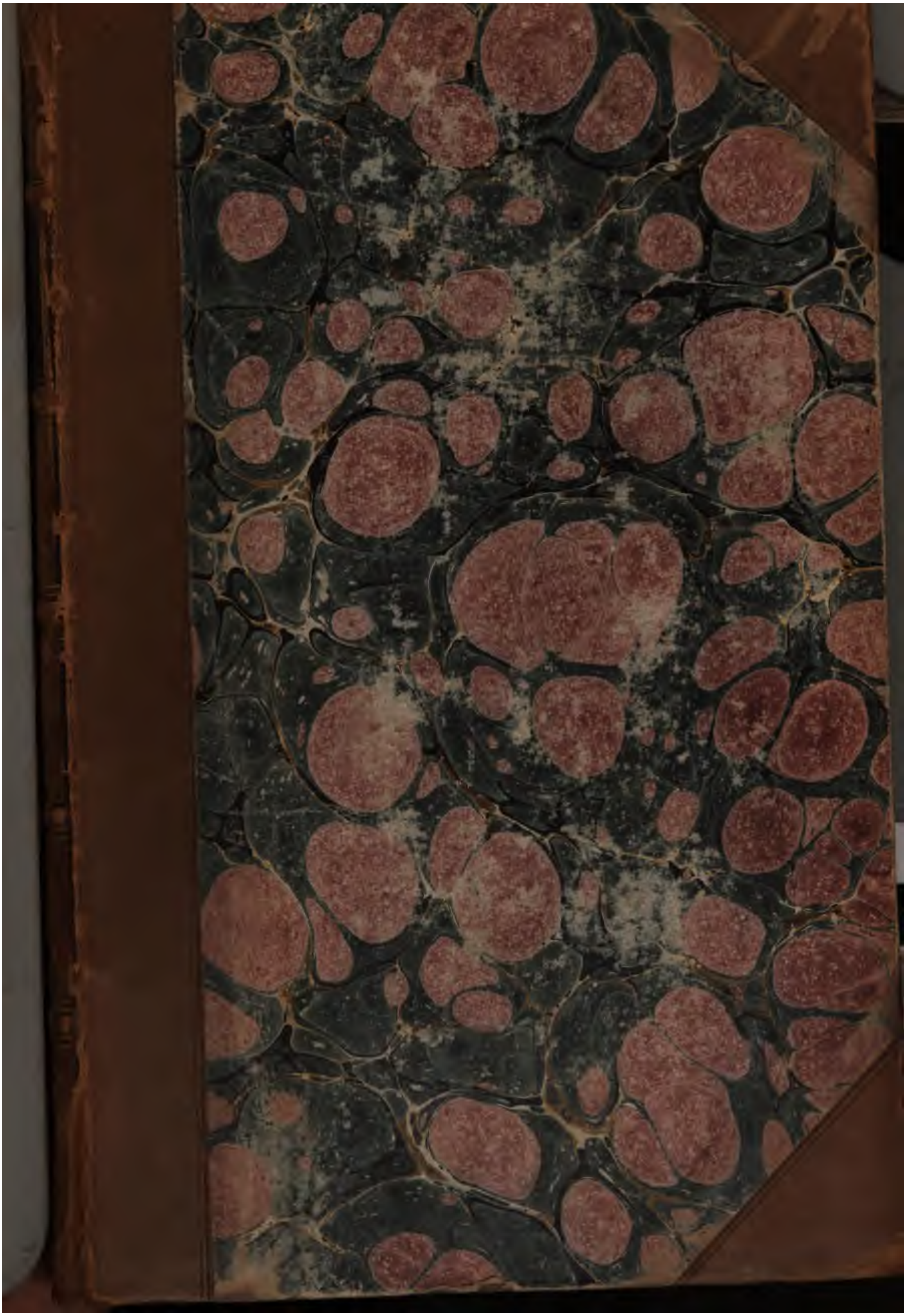

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PREFACE.

THROUGHOUT the course of the eighteenth century, a period in which the philosophy of nature has achieved its highest triumphs, the science which treats of the phenomena of light and of the laws of vision seems to have been almost overlooked. From the era of Newton's discoveries, for more than a hundred years, few important contributions were made to the knowledge of the physical properties of light; while the mathematical development of the laws of reflexion and refraction, and its application to the theory and construction of telescopes, engaged, as compared with the other mixed sciences, little notice. The names of Euler, indeed, and of Clairaut, of D'Alembert, and Boscovich, are to be found enrolled in the history of this department of science; but owing to the painful and elaborate analysis which they have employed, and the complicated and inelegant character of the results which have in general arisen from its application, the researches of these writers, even where well directed, have been little known.

At length, however, these important subjects have begun to receive their merited share of attention. The discoveries of Young and Malus have opened a new and fertile field in experimental philosophy; nor have the labourers been unskilful or the harvest scanty. The beautiful and diversified phenomena of polarized light, which were developed during the researches that followed, have riveted the attention of the scientific world for the last twenty years; and the singular connexions that have been found to subsist between these phenomena and the internal structure of the bodies by whose action they are exhibited seem to promise a clue to some of the most mysterious recesses of the labyrinth of nature. Of late years, too, the mathematical branches of optics have received a fresh and vigorous impulse from the hands of some of the ablest analysts of this country; and Science has pressed forward to lend her all-important aid to Art in the improvement of those instruments which have extended the domain of sense, and borne the mind of man into those realms of space in which the imagination itself scarcely dares to wander.

The following pages were commenced in the hope that they might prove in some degree instrumental in facilitating the approach to this interesting region of science. In their progress the author has had recourse to most of the sources from which he expected to derive information and assistance. Of these aids, however, he has availed himself rather in pointing out the results, than in directing the steps by which they were to be attained; and it is hoped that any sacrifice which he may have been compelled to make in this respect, will be more than compensated by the advantages attending uniform and analogous methods.

Much benefit too, it is hoped, has arisen from the peculiar notation which has been adopted throughout the greater part of this volume. In this notation, which has been employed by Mr. Herschel in his valuable *Essay on Light*, the reciprocal of the focal distance is considered as a distinct variable, and denoted by a proper symbol. Nor is this function of importance merely as a subsidiary quantity, introduced for the purpose of facilitating analysis and giving symmetry to its results: it has a real and positive signification, being the measure of the degree of divergence or convergence of the pencil. The author has accordingly ventured upon the use of a new term, and designated it the *vergency* of the pencil.

The present volume is divided into three parts. The first of these contains the theory of simple or homogeneous light, and its several modifications in direct transmission, reflexion, or refraction. In the general theory of reflexion and refraction, as laid down in the fourth and seventh chapters of this part, surfaces of revolution alone have been considered, and the subject confined to the case in which the incident ray is in a plane passing through the axis. This case, which is in itself sufficiently comprehensive, and beyond which it is necessary to have recourse to the geometry of three dimensions, seems to present a natural resting-place in the boundless field of investigation; and it is accordingly the limit which the author has proposed to himself in these pages as best suited to the purposes for which they were written. The reader who is prepared to grapple with the difficulties of the subject in its most comprehensive form is referred to the memoir of Professor Hamilton, entitled *An Essay on the Theory of Systems of Rays*, published in the Transactions of the Royal Irish Academy—one of the ablest efforts of analytical skill of which the present day can boast. Many of the most important cases, not included within the limit above mentioned, will also be found in Professor Airy's paper *On the Spherical Aberration of the Eye-pieces of Telescopes*, published in the Transactions of the Cambridge Philosophical Society, and in Mr. Coddington's late *Treatise on the Reflexion and Refraction of Light.*

The second part of this volume contains the theory of compound or solar light—the discoveries of Newton, Dollond, Blair, and Fraunhofer—the laws of the dispersion of light by any combination of prisms or lenses, and the conditions of achromatism—and, finally, the explanation of the phenomena of the colours of natural bodies.

The third part treats of the laws of vision—the human eye—and the various combinations of lenses or specula by which it is aided in the perception of minute or distant objects. In this part it has been the author's aim to introduce the scientific reader to a knowledge of the general theory of telescopes and microscopes, rather than to direct the artist in their construction; and accordingly the subject, though somewhat fully considered, has not been pursued into the details requisite to the latter purpose. The constructions of some of the simpler optical instruments —such as the camera lucida, the camera obscura, the magic lantern, &c.—which do not strictly come under the head of assisted vision, are described in the earlier chapters of the work, whose principles they tend to illustrate.

In the appendix the principles already laid down are applied to explain the laws of atmospheric refraction, and to account for some other phenomena—as rainbows, halos, &c.—connected with the atmosphere, and arising from the reflexion and refraction of light.

Should these pages be found of service to those for whose use they are destined, it is the writer's hope that he may be enabled, at some future period, to complete the task which he has undertaken by the publication of a second volume on physical optics. Of the future, however, he is compelled to speak vaguely. The laborious duties of the profession to which he belongs may conspire with other causes, over which he has no control, to withhold him from again appearing before the public; and the delays of various kinds, which the present volume has had to experience in its progress through the press, have deducted largely from the stock of self-reliance with which it was commenced.

TRINITY COLLEGE,
Feb. 15, 1831.

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ERRATA.

Page 60, line 6 from bottom, *for* (77.), *read* (75.).
66, 5 from bottom, *for* $(1 + p\mathrm{x}^2)$, *read* $(1 + p^2)\mathrm{x}$.
67, 7, *insert* q as a multiplier of the first member of the equation.
123, 16, *for* $^2\theta^2$, *read* $\mu^2\theta^2$.
126, 17, *insert* φ after sign of equality.
176, 12, *for* $\left(\frac{d}{dx}\right)^2$ *read* $\left(\frac{d\varrho}{dx}\right)^2$.
304, 5 from bottom, *insert* λ at beginning of line.
321, 12, *for* Kliengstierna, *read* Klingenstierna.
351, 14, *for* a, *read* $\frac{1}{a}$.
352, 4 from bottom, *after* unity, *insert* and no light being supposed to be absorbed in its passage through the lenses.
353, 4, *for* M, *read* ϱ.
372, 12, *insert* Θ before the sign of equality.

A TREATISE ON OPTICS.

PART I.

OF SIMPLE OR HOMOGENEOUS LIGHT.

INTRODUCTION.

OF THE NATURE AND GENERAL PROPERTIES OF LIGHT.

(1.) THE physical nature of light, or the immediate cause of the sensation which we call light, has not been, as yet, satisfactorily determined. Notwithstanding our ignorance on this fundamental point, however, the information which we actually possess concerning light is of the utmost value; by the aid of experiment and observation, philosophers have attained the knowledge of several of its leading properties; and it is the province of optical science to unfold these general properties, and by means of them to exhibit and explain all the less obvious modifications which it undergoes. Thus the science of optics is built upon certain leading facts, established by experience, and independent of all hypothesis whatsoever; and accordingly, the conclusions to which it leads us remain unshaken, whatever opinion respecting the nature of light itself shall eventually be found to be the true one.

(2.) *Light consists of separable and independent parts.*

This property, which is assumed in all our reasonings respecting light, is readily established by familiar experience. Any portion of light may be intercepted by an opaque obstacle, and the rest allowed to pass; and this latter part is never found to be affected in any way by its separation from that which is intercepted.

The smallest portion of light which we can thus intercept, or allow to pass, is called a *ray*.

(3.) *The direction of the rays of light, while they continue in the same medium**, *is always rectilinear.*

This may be familiarly exhibited by suffering a beam of the sun's light to enter a dark room through a small aperture; the particles of dust or smoke, floating in the atmosphere of the room, reflect the light of the sun, and exhibit the form of the beam, which is always observed to be rectilinear. 2dly, It appears also from the fact, that it is impossible to see any object when an opaque body is interposed in the right line joining the object and the eye. 3dly, It is also proved by the shadows of bodies, which are always bounded by right lines.

(4.) What has been hitherto said applies to light, in whatever state it is found. Light, however, undergoes various modifications under different circumstances; and these modifications are found to be subject to general laws, which are therefore to be ranked amongst its leading properties.

The modifications of light, with which we have to do at present, are chiefly three: 1. When propagated directly from some object which has in itself the power of exciting the sensation of light in the eye: such bodies are termed *luminous*, and the light proceeding from them is said to be *direct* or *emitted*. 2. When meeting with any obstacle it is turned back into the medium from which it came: it is then said to be *reflected*. 3. When it passes through the medium which it meets, but bent into a different course: it is then said to be *refracted*.

(5.) *Light is propagated from luminous bodies in all directions, in straight lines.*

The former part of this proposition is evident from the circumstance that luminous bodies are visible, whatever be the position of the eye, provided no body intervenes to intercept their light. The latter part of the proposition has been already established.

This may be considered as the fundamental law of *direct* or *emitted light*.

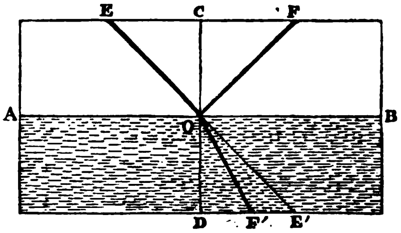
---

* Any space which light traverses, whether occupied by matter or not, is termed a *medium*.

(6.) When a ray of light falls upon the surface of any medium, and is there reflected, the portions of the ray, before and after the incidence on the medium, are called the *incident* and *reflected rays*, respectively; and the angles, which they form with the perpendicular to the surface at the point of incidence, are called the *angles of incidence* and *reflexion.*

The angles of incidence and reflexion are always in the same plane, and equal.

This is the fundamental law of *reflected light*, and may be exhibited by the following simple experiment:—Let a rectangular piece of wood or pasteboard be provided, having its opposite sides bisected by the lines AB, CD; and let it be immersed perpendicularly in water, as far as the line AB, and so placed that a small beam of the sun's light, admitted through the shutter into a darkened room, may be incident at the point O, along the line EO. A portion of this ray will be observed to be turned back, or *reflected,* along the line OF, on the same surface; and the distances CE, CF, being measured, are found equal.



Now, since the incident and reflected rays, EO, FO, together with the perpendicular to the reflecting surface, CO, all lie in the plane of the board, it follows that the angles of incidence and reflexion are in the same plane. And since the distances CE, CF, are found equal,—CO being common to the two triangles COE, COF, and the angles at C right,—it follows that the angles at O, or the angles of incidence and reflexion, are also equal.

This experiment should be repeated at different altitudes of the sun, so as to give different incidences.

(7.) When a ray of light falls upon the surface of any medium, and is there refracted, the two portions of the ray, before and after incidence on the medium, are called the *incident* and *refracted rays;* and the angles, which they make with the perpendicular to the surface at the point of incidence, are called the *angles of incidence* and *refraction*, respectively.

The angles of incidence and refraction lie always in the same plane, and their sines are to one another in an invariable ratio.

This is the fundamental law of refracted light, and may be illustrated in the following manner:—All things being arranged as in the last experiment, it will be found, as before, that part of the incident light is turned back or reflected at the surface of the water; part, however, will enter the water, and this portion, instead of proceeding directly forward in the line OE′, is observed to be bent into the line OF′, on the same surface.

Now, since the incident and refracted rays, EO, OF′, together with the perpendicular to the refracting surface, CD, all lie in the plane of the board, it follows that the angles of incidence and refraction are in the same plane. Also, if the distances CE, DF′, be measured, and thence EO, F′O, computed, it will be found that the fractions $\frac{EC}{EO}$, and $\frac{F'D}{F'O}$, always bear to one another a constant ratio, whatever be the direction of the incident ray EO; *i. e.* that the sines of the angles EOC, F′OD, (which are the angles of incidence and refraction) are to one another in an invariable ratio.

(8.) *Light is propagated, in free space, with a velocity of about* 195,000 *miles per second.*

That the propagation of light is not instantaneous, but progressive, was first discovered by Roemer, while engaged in observing the eclipses of Jupiter's satellites. He found that when Jupiter was in *opposition*, and therefore at a distance from the earth less than his mean distance by the radius of the earth's orbit, the eclipses happened about eight minutes *earlier* than they should according to the astronomical tables; while, when Jupiter was in *conjunction*, and therefore farther than his mean distance by the radius of the earth's orbit, these eclipses happened eight minutes *later*. Hence, supposing these differences to arise from the progressive propagation of light, its velocity must be such that it shall traverse the diameter of the earth's orbit in sixteen minutes nearly; and therefore in a second it describes about 195,000 miles.

The progressive propagation of light was afterwards most successfully applied by Bradley to explain the phenomenon of the *aberration* of the fixed stars. From the theory of aberration, thus explained, it appears that the velocity of the light of the fixed stars is to the velocity of the earth in its orbit as radius to the sine of the greatest aberration, which is about $20\frac{1}{4}''$. Now the value of the velocity of light, obtained in this way, agrees with that deduced from the eclipses of Jupiter's satellites, within limits which may be fairly ascribed to the errors of observation; the difference corresponding to a variation in the observed quantities, which is below the probable amount of such errors. Hence, these results, derived from sources so widely different, may be considered as completely confirming each other.

From this we conclude, also, that the direct light of the fixed stars, and the reflected light of the satellites, proceed with the same velocity.

(9.) Before we conclude this part of our subject, it may not be amiss to give a brief sketch of the two principal hypotheses, which have, in latter times, been held respecting the physical nature of light.

The Cartesians held that light consisted in the *undulations* or pulses of a highly elastic medium, which pervades all space, and to which the oscillatory motion is communicated by the impulse of some body, which is therefore called luminous; in the same manner as sound is known to consist in the pulses of the air, which is set in vibration by the impulse of the sounding body. This elastic medium is conceived to be of such extreme tenuity, as to afford no appreciable resistance to the motions of the planets, comets, &c. It also penetrates all bodies, but is supposed to be of a different density and elasticity within them than when free and disengaged; and it is on this difference in its state that the reflexions and refractions, which light undergoes in encountering such bodies, are made to depend.

Newton, on the contrary, maintained that light is a *material substance*, the minute particles of which are projected or emitted from the luminous body, with a prodigious velocity, in all directions. These minute bodies are supposed to be acted on by attractive and repulsive forces, resident in the bodies which they meet, and thus to be turned out of their course according to the laws of reflexion and refraction.

The Newtonian hypothesis has been that generally received by philosophers since its publication by the great inventor. That of Des Cartes, however, has not been without its supporters; the celebrated Huygens has deduced from it all the leading phenomena of light in his elegant treatise *De la Lumiere;* and in the present day it has had two able supporters in M. Fresnel and the lamented Dr. Young. It is somewhat remarkable, that all the principal phenomena of light are deducible, by strict mathematical reasoning, from either supposition; so that there does not appear, on a general view at least, to be any *experimentum crucis*, to which we can appeal, to establish the truth or falsehood of either*.

(10.) The principal objection urged against the *undulatory hypothesis* is that, according to it, light should not only be propagated in right lines, but in every direction, as is the case with sound; so that there could be no shadow, no absence of light produced by the intervention of an obstacle. To this the supporters of the system reply, that this lateral propagation of the pulses of an elastic medium is less, the greater the velocity of the pulses, as that of sound is less than that of water, &c.; so that they consider themselves justified in inferring, that when the velocity of propagation is so great as that of light, the lateral propagation must be insensible.

Against the *material hypothesis* have been urged,

1st, The waste of the luminous bodies, which would arise from emission, if light be a material substance; and,

2dly, The interference of the rays, and the consequent disturbance of vision, which would result from the meeting of the particles of light, flowing from innumerable luminous objects in every direction.

In answer to the first of these objections, it is asserted, that the particles of light are of such extreme minuteness, that no sensible diminution could thence arise, in any finite space of time, in the magnitude of the luminous body. The second objection is entirely obviated by showing, that it is by no means necessary that the particles constituting a ray of light should be contiguous, nor therefore that the rays should interfere in crossing. It appears by experience, that the impression produced by light on the retina continues about $7'''$ after the light has ceased to act, and in this time light describes more than 22,000 miles; wherefore, if the distance between any two particles which compose a ray be not greater than this quantity, the sensation produced by them will be continuous. Accordingly, we may suppose a space of 22,000 miles to intervene between the particles composing a ray, a space far more than sufficient to prevent interference.

---

* In the explanation of the law of refraction, the *undulatory hypothesis* requires that the velocity of the light should be less in the denser medium than in the rarer; while, in the *material system,* the contrary must be the case. Could this fact be submitted to experiment, we should immediately be enabled to decide the question.

The Newtonian theory, however, is not without its difficulties. The nicest experiments have failed to exhibit any of the properties of matter in light: the light of the sun has been condensed into a focus by a powerful burning glass, and thus made to impinge upon a thin metallic leaf, carefully suspended *in vacuo;* and though the substance transmitted few or none of the incident rays, not the slightest motion could be detected. Further, the velocity of light itself—a velocity which would require an impulsive force inconceivably great to emit the particles—seems to make against the material hypothesis. And, lastly, the *uniformity* of this velocity—proceeding, as light does, from such various bodies as those composing the system of the universe, and situated at such various distances—seems hardly reconcileable to any system which makes that velocity depend upon an emissive force.

On the whole, the question is still a doubtful one. But, whichsoever way it be decided, the truths of optical science remain unmoved; for these truths are the necessary consequences of the "laws of optics," which, as has been already stated, are facts derived from observation and experience, and independent of all hypothesis. Thus, the question respecting the nature of light, however interesting in itself, is, as far as relates to these conclusions, one of a nature merely speculative.

CHAPTER I.

OF DIRECT LIGHT.

I.

Of the Intensity of Direct Light at different Distances.

(11.) WHEN a medium transmits the whole of the light which is incident upon it, it is said to be *perfectly transparent*, or *free;* and, on the other hand, it is said to be *imperfectly transparent*, when it transmits only a part of the incident light and intercepts the rest.

(12.) *In a free medium, the intensity of the light, which is propagated in parallel rays, is constant.*

This is evident: for, in a free medium, there is no obstacle whatever to the progress of the light; and the rays, being parallel, always preserve the same mutual distance.

(13.) *In a free medium, the intensity of the light, which is propagated in rays diverging from a point, varies inversely as the square of the distance from that point.*

For if we conceive several concentric spherical surfaces, having the given point as their common centre, severally to receive the light diverging from that point; it is evident that the *whole quantity* of light, incident on each of these surfaces, must be the same; and, accordingly, the *intensity* of the light on each will be inversely as the space over which it is diffused, *i. e.* inversely as the surfaces themselves, or the squares of their radii, which are the distances from the radiating point. Wherefore, if A be taken to represent the intensity of the light at the unit of distance, that at any other distance, δ, will be represented by the fraction

$$\frac{A}{\delta^2}.$$

(14.) The intensities of two luminous points being given, it is required to determine, in the line connecting them, the point of equal illumination.

Let m and n be taken to represent the intensities of the two lights at the unit of distance; also, let a denote the interval between them, and x the distance of the required point from the light m. Now, the intensities of the two lights at this point are $\frac{m}{x^2}$ and $\frac{n}{(a-x)^2}$ respectively; wherefore, by the conditions of the question, there is

$$\frac{m}{x^2} = \frac{n}{(a-x)^2}, \text{ or } \frac{\sqrt{m}}{x} = \frac{\pm\sqrt{n}}{a-x};$$

from which we get the twofold value of x,

$$x = \frac{a\sqrt{m}}{\sqrt{m} \pm \sqrt{n}},$$

from which it appears that there are two points, in the line joining the two lights, at which their illuminations are equal.

When $m = n$, or the two lights of equal intensity, one of the values of x becomes infinite, and the other is reduced to $\frac{1}{2}a$; showing that, in this case, the point of equal illumination is in the middle of the line joining the two lights, or infinitely distant from both.

(15.) The intensities of two luminous points being given, it is required to determine the equation of the surface, every point of which is equally illuminated by the two lights.

m and n denoting the intensities of the two lights (as before), and a the interval between them, let the joining line be taken as the *axis of abscissæ*, and one of its extremities as the *origin;* then, at any point whose co-ordinates are x, y, z, the intensity of the light proceeding from the two points will be, respectively,

$$\frac{m}{x^2+y^2+z^2} \text{ and } \frac{n}{(a-x)^2+y^2+z^2};$$

therefore, equating these values, and reducing, we have for the equation of the required surface,

$$x^2+y^2+z^2+\frac{2ma}{n-m}.x = \frac{m}{n-m}.a^2;$$

and if we transform this equation, by making $x = x' - \frac{ma}{n-m}$, it becomes

$$x'^2 + y^2 + z^2 = \frac{mn \, . \, a^2}{(n - m)^2},$$

the equation of a *sphere*, whose radius $= \frac{\sqrt{mn}}{n - m} \, . \, a.$

The centre of this sphere is at the origin of the transformed co-ordinates; it lies, therefore, on the line joining the luminous points, and at a distance from the former, $= \frac{ma}{n - m}.$

If we seek the intersections of this sphere with the line joining the two points, by making $y = 0$, $z = 0$, we find, as before,

$$x = \frac{a\sqrt{m}}{\sqrt{m} \pm \sqrt{n}}.$$

When $m = n$, or the intensities of the two lights equal, the radius of the sphere becomes infinite, as does also the abscissa of the centre; indicating that the sphere becomes, in this case, a *plane*. In fact, the former of the equations, which we obtained above, becomes in this case $2x = a$; indicating a plane perpendicular to the joining line, and cutting it at its middle point.

(16.) If, now, the light diverging from the luminous point be received upon any object, the degree of illumination of each point of this object will be, *cæt. par.*, proportional to the quantity of light incident on the unit of surface. If then the form of the object be a spherical surface, whose centre is the luminous point, as every portion of that surface is perpendicular to the incident light and equidistant from the luminary, the quantity of light incident upon the unit of surface will be the same throughout the whole extent of the object, and equal to $\frac{\mathrm{A}}{\delta^2}$, δ being the radius of the spherical surface, or the distance of the object. Hence, if ϱ be a constant quantity, whose magnitude is dependent on the nature of the object, the degree of illumination is represented by

$$\frac{\mathrm{A}\varrho}{\delta^2}.$$

The same formula will also represent the degree of illumination of a small plane object, placed so as to receive the incident light perpendicularly; since such an object may, without error, be considered as a portion of a spherical surface concentric with the luminary.

If the illuminated object be a small plane surface inclined to the incident light, the quantity of light incident upon the unit of surface will be diminished in the proportion of radius to the sine of the inclination of the surface to the line connecting it with the luminous object. Hence, if θ be taken to denote this angle, the degree of illumination in this case is represented by

$$\frac{A\varrho . \sin. \theta}{\delta^2}.$$

When the magnitude of the illuminated object is not inconsiderable, the angle of inclination and the distance both vary, and the degree of illumination will vary throughout the extent of the object, according to the formula just given.

(17.) A small white surface being placed horizontally upon a table, and illuminated by a lamp or candle, placed at a given horizontal distance, a, required the height of the flame which will give the greatest possible illumination.

Taking θ as the variable, there is $\delta = \frac{a}{\cos. \theta}$, and the degree of illumination is equal to

$$\frac{A\varrho}{a^2} . \sin. \theta . \cos.^2 \theta.$$

Wherefore, making $\sin. \theta \cos.^2 \theta$ a maximum, we find

$$\cos.^2 \theta - 2 \sin.^2 \theta = 0,$$

$$\text{or } \tan.^2 \theta = \tfrac{1}{2}.$$

Now, if h denote the height of the flame, $h = a \tan. \theta$; wherefore, in the case of greatest illumination,

$$h = \frac{a}{\sqrt{2}}.$$

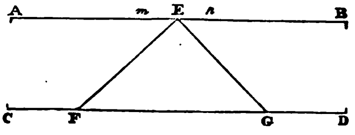
(18.) The eye affords no *direct* means of measuring the intensity of any light, or of estimating the relative strength of different lights compared together; for, though capable at once of discerning a *difference* in the intensities of two unequal lights, presented to it at the same time, it is wholly unable to determine their comparative strength. And again, one and the same degree of light will affect the eye, at different times, with widely different impressions, dependent on the state of sensibility of the organ, which is continually changing. By the aid of that power which we possess, however, namely, of detecting *a difference*, when it exists, in the intensities of two lights presented together, we are enabled, indirectly, to compare the relative intensities of any two lights.

Let two pieces of white paper, of the same size and material, and placed close together, be illuminated each by one of the two lights whose intensities we would compare, the light being made to fall on them perpendicularly. Then, one of the lights being shifted backwards and forwards, until there is no longer any sensible difference in the degree of illumination of the two objects, the intensities of the two lights will be obviously as the squares of their respective distances from the objects illuminated.

If the lights to be measured be not readily moveable, the equalization may be effected by inclining the illuminated objects at various angles to the incident light, and noting the angles when the illuminations are equal; the intensities of the two lights, then, will be as the squares of the distances directly and inversely as the sines of the angles of inclination.

(19.) A very convenient method of employing the principle described in the last article was suggested by Count Rumford. It consists in equalizing the *shadows* cast by the two lights to be compared on a screen or surface prepared to receive them. The shadow cast by each light, in this case, is illuminated by the other, and the rest of the screen by both lights together; if, then, the shadows be of equal intensity, the illuminations of the two lights must be so too, and the intensities of the lights themselves, therefore, as the squares of their distances from the screen.

(20.) Mr. Ritchie's *photometer* is also founded on the same principle. It consists of a rectangular box, ABCD, open at the opposite sides, AC, BD. In the centre of the top of the box is a narrow slit, *mn*, covered with oiled paper; and within are fixed two rectangular pieces of plane looking-glass, EF and EG, with their reflecting surfaces turned outwards, inclined to the top, each at an angle of 45°, and meeting in E, the middle of the slit; where



also is fixed a strip of blackened card, to prevent the lights reflected from the two glasses from intermingling.

In using this instrument, the box is to be placed between the two lights to be compared, so that the light of each shall fall on the reflector next to it, and then moved towards one or other of the lights, until the paper *mn* appears equally illuminated on both sides of the division E. The intensities of the two lights then, it is evident, are as the squares of their distances from the centre of the box.

(21.) We now proceed to consider the variation of the intensity of light in passing through a medium of *imperfect transparency*.

First, then, when the light is propagated in parallel rays: let A denote the absolute intensity of the light, or the whole number of rays, at its entrance into the medium; and α the ratio which the transmitted part bears to the whole, after passing through a unit of thickness; then $A\alpha$ is the portion transmitted through the first unit; and, since α is a constant quantity whose magnitude is independent of the quantity of the incident light, the portion transmitted through the second unit of thickness will be $A\alpha^2$; that transmitted through three such units, $A\alpha^3$; and, finally, the portion transmitted through any thickness, denoted by θ, will be

$$A\alpha^\theta,$$

in which α is a quantity less than unit, whose value depends upon the nature of the medium.

Hence it appears, that as the thickness increases in arithmetical progression, the intensity of the transmitted light, which is propagated in parallel rays, is diminished in geometric progression.

From this it follows that the transmitted light can never be wholly extinguished by a medium of any finite thickness; for, however small the quantity α be, the quantity $A\alpha^\theta$ can never vanish for any finite value of θ, however great. When, however, the fraction α is small, its power, α^θ, becomes so very small for a moderate value of the thickness θ, that the quantity of the transmitted light, $A\alpha^\theta$, may be regarded as wholly insensible.

(22.) This is the principle of the *lucigrade*, an instrument invented by François Marie, for the purpose of comparing the intensities of two lights together. The instrument consists merely of a number of glass plates, homogeneous and of equal thickness, placed together. Each light is observed through these plates, and plate after plate is added until it ceases to be visible; the ratio of the intensities is then determined by the number of the plates. For, let A and A' be the absolute intensities of the two lights, α the ratio which the transmitted portion bears to the whole after passing through a single plate, n and n' the number of plates added, in each case, until the light ceases to be visible. Then it is evident that the portions of the two lights which reach the eye under these circumstances are, respectively, $A\alpha^n$ and $A'\alpha^{n'}$. Now these, being each reduced to the limit of sensible light, are equal; we have, therefore,

$$A\alpha^n = A'\alpha^{n'}; \text{ whence, } \frac{A}{A'} = \alpha^{n'-n}.$$

(23.) When the light, diverging from a luminous point, is incident upon an imperfectly transparent medium, the law of the variation of its intensity is somewhat more complicated. Let δ be the distance of the luminous point from the surface of the medium, which we shall suppose to be a portion of a spherical surface whose centre is the luminary; then, if A denote the intensity of the light at the unit of distance, at its incidence upon the surface of the medium its intensity is $\frac{A}{\delta^2}$. Now, to obtain the intensity of the light after passing through a unit of the thickness of this medium, we must multiply this fraction by α (21), and substitute $\delta + 1$ for δ, so that it is $\frac{A\alpha}{(\delta + 1)^2}$; similarly, the intensity of the light after passing through two such units will be $\frac{A\alpha^2}{(\delta + 2)^2}$, and generally, it is expressed by

$$\frac{A\alpha^\theta}{(\delta + \theta)^2},$$

θ denoting the number of units of thickness of the medium which it traverses.

When the medium, through which the light is transmitted, is of a highly transparent nature, α is nearly equal to unit, and its powers decrease very slowly; in this case, therefore, the variation of the intensity will depend chiefly on the denominator of the fraction. On the contrary, when the medium stops a considerable portion of the light, the thickness through which any sensible portion of light is transmitted is inconsiderable; θ therefore may be neglected in comparison with δ, and the intensity is expressed by

$$A' a^{\theta},$$

in which $A' = \frac{A}{\delta^2}$ = intensity of light at its incidence on the medium of imperfect transparency.

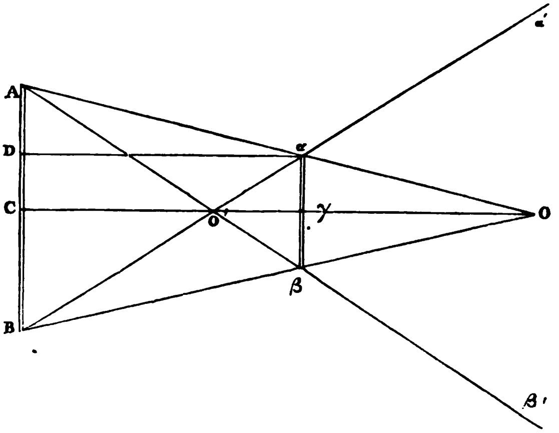
II.

Of Shadow and Penumbra.

(24.) We have hitherto considered light as proceeding in a medium, which affords either no obstruction whatever to its passage, or else produces a *gradual* diminution in its intensity. We are now to consider it as meeting with some obstacle which *perfectly* debars its progress; and to examine the total and partial absence of light thence arising.

To simplify the subject, we shall at first consider the luminous object, as well as the body which obstructs the passage of the light, as circular, and their planes perpendicular to the line joining their centres. Then AB, being the diameter of the luminous object, and $\alpha\beta$ that of the obstacle, let the lines Aα, Bβ, be drawn touching them externally, and Aβ, Bα, touching internally. Then it is evident that the space $\alpha o \beta$, contained by the external tangents, is *perfect shadow;* while in the angular spaces $o\alpha\alpha'$, $o\beta\beta'$, contained between the external and internal tangents, the light is but partially excluded, gradually diminishing from the internal to the external tangent; this space is called the *penumbra*, or partial shadow.

(25.) Let it be required to determine the heights of the conical shadow and penumbra, measured from the centre of the obstacle.



If the line αD be drawn parallel to the axis of the cones CO, on account of the similarity of the triangles $A\alpha D$, $\alpha o \gamma$, there is $\frac{AD}{\alpha D} = \frac{\alpha\gamma}{o\gamma}$; or, denoting by r and r' the semidiameters of the luminary and obstacle, the distance between them by δ, and the heights of the conical shadow and penumbra by h and h',

$$\frac{r - r'}{\delta} = \frac{r'}{h}, \text{ whence } h = \frac{\delta . r'}{r - r'}.$$

Again, by reason of the similar triangles $B\alpha D$, $\alpha o'\gamma$, $\frac{BD}{\alpha D} = \frac{\alpha\gamma}{o'\gamma}$; that is,

$$\frac{r + r'}{\delta} = \frac{r'}{h'}, \text{ whence } h' = \frac{\delta . r'}{r + r'}.$$

(26.) These results may be expressed under a different form; for, if the semiangles of the conical shadow and penumbra be denoted by ϵ and ϵ', then is

$$\frac{r'}{h} = \tan. \epsilon, \frac{r'}{h'} = \tan. \epsilon'.$$

And again, if θ denote the apparent semidiameter of the luminous object, as seen from the centre of the obstacle, θ' that of obstacle as seen from luminary, there is

$$\frac{r}{\delta} = \tan. \theta, \frac{r'}{\delta} = \tan. \theta';$$

if then these values be substituted in the equations of the preceding article, we obtain

$$\tan.\epsilon = \tan.\theta - \tan.\theta', \quad \tan.\epsilon' = \tan.\theta + \tan.\theta',$$

equations which determine the angles of the conical shadow and penumbra, and therefore also their heights measured from the obstacle.

If these equations be added together, we find

$$\tan.\epsilon + \tan.\epsilon' = 2\tan.\theta,$$

from which it appears that the tangent of the apparent semidiameter of the luminary is an arithmetical mean between the tangents of the semiangles of the two cones.

(27.) When the luminous object is infinitely distant, as in the case of the sun and moon, $\theta' = 0$, and therefore

$$\epsilon = \epsilon' = \theta.$$

Also, the angle of the penumbra, $\alpha'\alpha o$, which is equal to $\epsilon + \epsilon'$, in this case becomes 2θ. Hence, the angles formed by the lines bounding the shadow and penumbra, respectively, as well as the angular space occupied by the penumbra alone, are all equal to 2θ, the apparent diameter of the luminary.

In the case of the sun and moon, however, since 2θ is only about $30'$, the penumbra, as well as the diminution of the shadow arising from the convergence, will be hardly perceptible, unless the screen which receives it be at a considerable distance from the obstacle.

Also, since $h = r'.\cot.\epsilon$, $h' = r'.\cot.\epsilon'$, the heights of the conical shadow and penumbra, when the luminary is infinitely distant, are

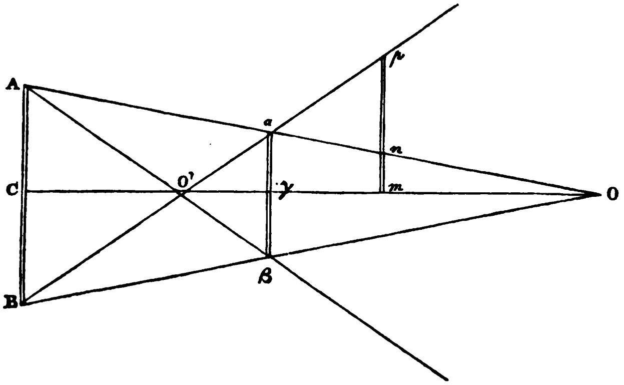
$$h = h' = r'.\cot.\theta.$$

(28.) A screen being placed to receive the shadow, it is required to find the breadth of the shadow and penumbra, at any distance from the obstacle.

mnp being the screen, whose plane is perpendicular to the axis of the shadow, let mn, the breadth of the perfect shadow, be denoted by b, mp, the breadth including penumbra, by b'; and let $m\gamma$, the distance of the screen from the obstacle, be x. Then, on account of the similar triangles $a\gamma o$, nmo, there is

$\frac{mn}{mo} = \frac{\gamma\alpha}{\gamma o}$, or $\frac{b}{h - x} = \frac{r'}{h}$; whence

$$b = r'\left(1 - \frac{x}{h}\right).$$



Likewise, in the similar triangles, $\alpha\gamma o'$, pmo', there is $\frac{mp}{mo'} = \frac{\gamma\alpha}{\gamma o'}$, or $\frac{b'}{h' + x} = \frac{r'}{h'}$; whence

$$b' = r'\left(1 + \frac{x}{h'}\right).$$

To find the breadth occupied by the penumbra alone, we have only to subtract the former of these expressions from the latter, and we find $b' - b = r'x\left(\frac{1}{h} + \frac{1}{h'}\right)$, but $\frac{r'}{h} + \frac{r'}{h'} = \tan.\,\iota + \tan.\,\iota' = 2\tan.\,\theta$, hence there is

$$b' - b = 2x\tan.\,\theta;$$

[illegible] which it appears that the breadth of the penumbra is [illegible] independent of the magnitude of the obstacle, and [illegible] only with the apparent diameter of the luminary and [illegible] of the screen from the obstacle—being equal to a [illegible], at the distance of the screen from the obstacle, subtends an angle equal to the apparent diameter of the luminous object.

(29.) It appears from the values of h and h', given (25.), that the latter is always positive; while the former is positive, negative, or infinite, according as r is greater, less, or equal to r'. It follows therefore, from the values of b and b', given in the preceding article, that b', the breadth of the entire shadow, always increases as the distance of the screen increases. When $r > r'$, b, the breadth of the perfect shadow, diminishes until $x = h$, when it vanishes altogether; when $r < r'$, b always increases with the distance; and, finally, when $r = r'$, b is invariable at all distances from the obstacle, being equal to r', the semidiameter of the obstacle. The breadth of the penumbra, $b' - b$, always increases with the distance of the screen.

(30.) What has been above established may be applied to the case in which the figures of the luminous object and obstacle are not circles, but any figures whatever, symmetrical with respect to a point; for we have only to consider the lines AB, $\alpha\beta$, as the intersections of any plane, passing through these points, with the plane of the two figures, and we shall thus have the properties of the section of the shadow and penumbra in that plane. It is obvious, however, that the lines bounding the shadow and penumbra will intersect all in the same point, only when the figures of the obstacle and luminous object, projected on planes perpendicular to the line joining their centres, are similar.

(31.) When the distance of the luminous object may be regarded as infinite, the shadow, projected on a screen parallel to and very near the obstacle, will be a figure similar to the obstacle; while, if the screen be removed to a considerable distance, the shadow will be similar to the luminous object. This will readily appear from the value of b', which in this case becomes

$$b' = r' + x \, . \, \frac{r}{\delta};$$

for, when x is very small, $b' = r'$, *q. p.*, and therefore the figure of the shadow is, as to sense, similar and equal to that of the obstacle. But, when x is very considerable, $b' = x \, . \, \frac{r}{\delta}$ nearly, and is therefore proportional to r, the semidiameter of the section of the luminary.

Hence is seen the reason why the shadows of bodies, projected by the sun's light, are circular when received on a screen at a sufficient distance.

(32.) It will be readily seen that what has been established relatively to light, a part of which is intercepted by an obstacle, will hold true also of light, part of which is suffered to pass through an aperture. We have only to substitute *perfect light* for perfect shadow, and to reverse the order of the decrease of light in the penumbra.

CHAPTER II.

OF LIGHT REFLECTED AT PLANE SURFACES.

I.

Of a Single Pencil of Rays reflected at Plane Surfaces.

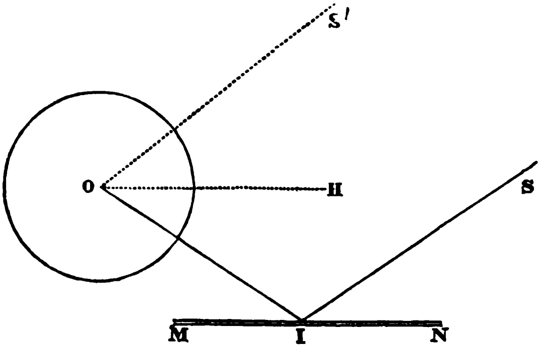
(33.) THE *angle of incidence* is the angle contained between the incident ray and the perpendicular to the surface at the point of incidence.

The *angle of reflexion* is the angle contained by the reflected ray with the same line.

The *angle of deviation* is the angle contained by the reflected ray with the incident ray produced.

(34.) It has been already shown that the angles of incidence and reflexion are in the same plane, and equal. As this law, however, is the foundation of the whole theory of reflected light, and as the method above given is suited to serve the purposes of illustration, rather than those of accurate demonstration, it will not be amiss to show how the equality of the angles of incidence and reflexion may be more strictly established.

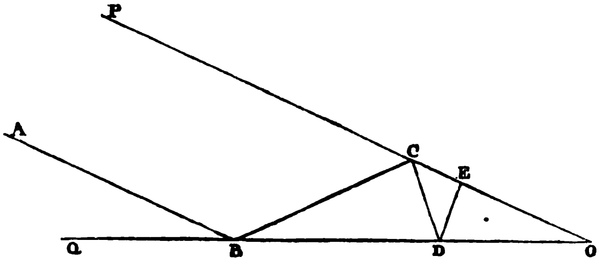
Let MN be any plane reflecting surface, placed horizontally. This position will be assumed by the surface itself, if it be fluid; if not, this position can readily be given to it by means of the spirit level. Let SI be a ray of light coming from some remote object, as the sun, and incident upon the surface at I; and let it be reflected in the direction IO. Now, let O, the centre of a small graduated circle, be placed somewhere in the line IO; and OS′, OH, being conceived drawn in the direction of the object and the horizon, let the angles S′OI and S′OH, the angular distances of the object from its reflected image, and from the horizon, respectively, be measured. It will be found that the former of these angles is always double the latter.



Now, the object being very remote, S′O may be considered as parallel to SI; also, OH is parallel to MN. Wherefore, if θ and θ' denote the angles which the incident and reflected rays make with the surface, there is S′OH = SIN = θ, HOI = OIM = θ', and therefore S′OI = $\theta + \theta'$; now S′OI is double S′OH, or $\theta + \theta' = 2\theta$, wherefore $\theta' = \theta$; and the angles of incidence and reflexion are equal.

(35.) If a ray of light meet a second reflecting surface, inclined to the first at any angle, it is required to determine its direction after any number of reflexions.

We shall limit ourselves to the case in which the reflexion takes place in a plane perpendicular to the two reflecting surfaces. Let OP, OQ, be the intersection of the two surfaces with



such a plane, and ABCDE the course of the ray after any number of reflexions. Then, if ι denote the inclination of the two mirrors; θ_1, θ_2, θ_3, &c. the angles formed by the ray with the surface at the 1st, 2d, 3d incidence, &c.; there is

$$\text{PCB} = \text{O} + \text{OBC} = \text{O} + \text{ABQ}, \text{ that is, } \theta_2 = \theta_1 + \epsilon.$$

In like manner we find $\theta_3 = \theta_2 + \epsilon$, &c. So that there is the following series of equations:

$$\begin{aligned} \theta_2 - \theta_1 &= \epsilon \\ \theta_3 - \theta_2 &= \epsilon \\ \cdots&\cdots \\ \theta_{n+1} - \theta_n &= \epsilon \end{aligned}$$

and, adding, there is

$$\theta_{n+1} - \theta_1 = n\epsilon.$$

When n is an even number, the angles θ_{n+1} and θ_1 are contained with the same surface; but the difference of the angles formed by two right lines with the same surface is evidently equal to the angle contained by the lines themselves; therefore the angle contained by the original ray, and that after n reflexions—or the total deviation—is equal to $n\epsilon$, n times the inclination of the mirrors.

When the ray suffers one reflexion at each mirror, the angle contained between the direct and the doubly reflected ray is double the inclination of the mirrors. This is the principle of Hadley's sextant, an instrument to be described hereafter.

(36.) From the formula of the last article it appears that the value of θ_{n+1} is increasing at each reflexion by ϵ, the inclination of the mirrors. When $\theta_1 + n\epsilon = \frac{\pi}{2}$, that is, if the first angle of incidence, $\frac{\pi}{2} - \theta_1$, be equal to $n\epsilon$, any multiple of ϵ, then $\theta_{n+1} = \frac{\pi}{2}$, or the ray after n reflexions will be perpendicular to one of the mirrors, and therefore return back exactly in the same course. When this is not the case, it is evident that some value of n will render $\theta_{n+1} = \theta_1 + n\epsilon$, greater than $\frac{\pi}{2}$; in which case the ray, after going up the angle, will return back, but in a different course. Finally, when θ_{n+1} is either equal to, or greater than π, the reflexion ceases; for the ray becomes either parallel to one of the mirrors, or meets it when produced backwards.

If the inclination of the mirrors be a submultiple of four right angles, *i. e.* if $n\iota = 2\pi$, then $\theta_{n+1} = 2\pi + \theta_1$; *i. e.* the ray after n reflexions will form with the surface the same angle as at first.

(37.) We now proceed to consider the reflexion of several rays, composing what is called in optics a pencil.

Any number of rays proceeding from, or tending to, a point, is called a *pencil of rays*.

When that point is infinitely distant, the rays constituting the pencil are *parallel*. In all other cases they are said to *diverge* or *converge*, according as they proceed from, or tend to, the common point.

The point from which the rays proceed, or to which they tend, is called the *focus* of the pencil; and the line drawn from this point, perpendicular to the reflecting or refracting surface, is called the *axis* of the pencil.

The focus of the incident pencil is sometimes called the *radiant;* and the focus of the reflected or refracted pencil, its *conjugate.*

These *foci* are said to be *real*, when the rays actually meet there; *imaginary*, or *virtual*, when they meet only when produced.

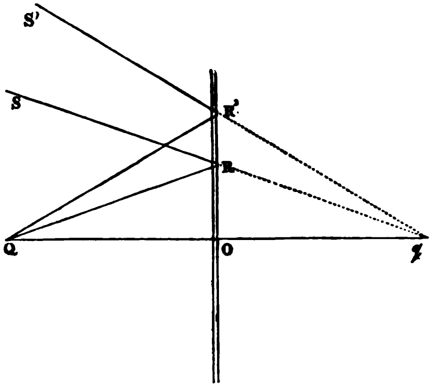
(38.) If a pencil of parallel rays be reflected at a plane surface, the reflected rays are also parallel.

For, since the incident rays are parallel, as also the perpendiculars to the surface at the points of incidence, the planes of reflexion must be parallel, and the angles of incidence equal. But the angles of reflexion are equal to the angles of incidence, and therefore equal to each other. Hence, as the reflected rays lie in parallel planes, and contain equal angles with parallel lines, namely, the perpendiculars to the reflecting surface, they must be parallel.

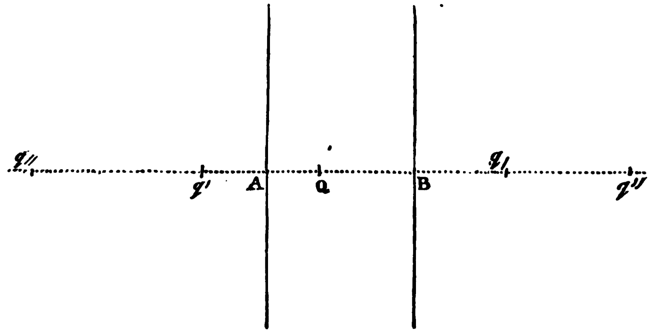
Hence, all that has been said above, respecting the direction of a single ray after any number of reflexions, may be also applied to a pencil of parallel rays.

(39.) If a pencil of rays, diverging from a point, be reflected at a plane surface, the foci of the incident and reflected rays will be equally distant from the surface, at opposite sides.

For, let QR be any ray proceeding from the point Q, the focus of the incident pencil, and let it be reflected in the direction RS. Again, let QO be a perpendicular from the point Q on the reflecting surface OR, and let it be produced to meet the reflected ray, produced backwards, at *q*. Then, since the incident and reflected rays make equal angles with the surface, $QRO = SRR' = qRO$; also RO is common to the two triangles, and the angles at O are right. Hence $qO = QO$. In the same manner, it will appear that any other reflected ray, R′S′, produced backwards, will meet the perpendicular QO in the same point *q*. That point is therefore the focus of the reflected pencil, and is, as has been proved, at the same distance behind the surface, as the point Q is before it.



(40.) A radiant point being situated between two parallel plane reflecting surfaces, required the foci of the reflected rays.



Through the point Q, the focus of incident rays, let the line AQB be drawn perpendicular to both surfaces, and produced indefinitely. Then taking $Aq' = AQ$, q' will be the focus of rays after reflexion by first surface. But these reflected rays will become incident on the second surface, diverging from q'. If, therefore, we take $Bq'' = Bq'$, q'' will be the focus of the rays after a second reflexion, and so forth. Again, the rays diverging from Q, and incident on the second surface, will have a conjugate focus in $q_{\prime}$; the rays diverging from this, and incident on first surface, have a conjugate focus in $q_{\prime\prime}$, $Aq_{\prime\prime}$ being equal $Aq_{\prime}$, &c. &c. Thus there are an infinite number of foci, all arranged on the line AB, and becoming more and more distant after each reflexion.

The distance Qq', Qq'', &c. are readily calculated.

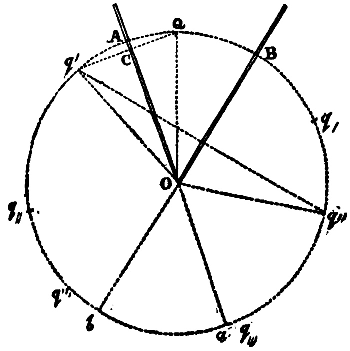
For, making $QA = a$, $QB = b$, and $AB = a + b = c$, there is

$$\begin{aligned} Qq' &= 2AQ = 2a, \\ Qq'' &= BQ + Bq' = Qq' + 2BQ = 2a + 2b = 2c, \\ Qq''' &= AQ + Aq'' = Qq'' + 2AQ = 2c + 2a, \\ Qq'''' &= \ldots\ldots\ldots\ldots\; 2c + 2a + 2b = 4c, \text{ \&c. \&c.} \end{aligned}$$

In like manner we find

$$Qq_{\prime} = 2b,\quad Qq_{\prime\prime} = 2c,\quad Qq_{\prime\prime\prime} = 2c + 2b,\quad Qq_{\prime\prime\prime\prime} = 4c, \text{ \&c.}$$

(41.) A radiant point being situated between two plane reflecting surfaces, inclined to one another at any angle, required the position and number of the foci of reflected rays.



OA and OB being the sections of the two surfaces by a plane perpendicular to both, and passing through Q, the focus of incident rays, let the perpendicular QC be let fall from this point on OA, one of the surfaces, and produced till $q'C = QC$, q' will be the focus after the first reflexion. Again, letting fall the perpendicular $q'q''$ from q' upon OB, and producing it equally to q'', q'' will be the focus of the rays after a second reflexion, &c. In like manner we find another series of foci, $q_{\prime}$, $q_{\prime\prime}$, $q_{\prime\prime\prime}$, &c. of which the first is produced by reflexion at the second surface.

Now, in the triangles QOC, q'OC, since q'C and QC are equal, OC common, and the angles at C right, $Oq' = OQ$; in the same manner $Oq'' = Oq' = OQ$. Therefore the foci all lie on the circumference of a circle, whose centre is the intersection of the sections of the mirrors, and radius the distance of the radiant point from the same.

To determine the places of the foci, let the arc QA be θ, $QB = \theta'$, and $AB = \theta + \theta' = \epsilon$.

Then $Qq' = 2QA = 2\theta$.

$$Qq'' = BQ + Bq' = Qq' + 2BQ = 2\theta + 2\theta' = 2\epsilon.$$

$$Qq''' = AQ + Aq'' = Qq'' + 2AQ = 2\epsilon + 2\theta, \text{ \&c.}$$

Similarly, $Qq_{\prime} = 2\theta'$, $Qq_{\prime\prime} = 2\epsilon$, $Qq_{\prime\prime\prime} = 2\epsilon + 2\theta'$, &c.

And, in general, the distances in the first series are,

$$Qq^{2n} = 2n\epsilon, \quad Qq^{2n+1} = 2n\epsilon + 2\theta.$$

In 2d. $Qq_{2n} = 2n\epsilon$, $Qq_{2n+1} = 2n\epsilon + 2\theta'$.

(42.) The number of images in this case is limited; for, when any of the images falls on the arc ab, *i. e.* between the mirrors produced, being behind both mirrors, no further reflexion can take place. Thus, if the image q^{2n} fall on the arc ab, then, observing that this image lies behind the second mirror, BO, there is the distance $Qq^{2n} > QBa$, *i. e.* $2n\epsilon > \pi - \theta$,

$$\text{or } 2n > \frac{\pi - \theta}{\epsilon}.$$

If the image which falls upon the space ab, be one of those behind the first mirror, or q^{2n+1}, there is $Qq^{2n+1} > QAb$,

$$i.\ e.\ 2n\epsilon + 2\theta > \pi - \theta', \text{ or } 2n\epsilon + \theta + \theta' > \pi - \theta,$$

$$\text{or } 2n + 1 > \frac{\pi - \theta}{\epsilon}.$$

The same result as before, $2n$ being the number of images in the former case, and $2n + 1$ in the latter. Therefore the number of images in the first series is the whole number next greater than $\frac{\pi - \theta}{\epsilon}$; and, in like manner, the number in the second series will be found to be the whole number next greater than $\frac{\pi - \theta'}{\epsilon}$.

If ϵ be a submultiple of two right angles, or $\frac{\pi}{\epsilon}$ a whole number, the number of images in each series will be $\frac{\pi}{\varepsilon}$, since $\frac{\theta}{\varepsilon}$ and $\frac{\theta'}{\varepsilon}$ are proper fractions. Therefore in this case the total number of images is $\frac{2\pi}{\varepsilon}$.

But in this case it happens that two images of the different series will coincide. For if $\frac{\pi}{\varepsilon}$ be an even number, or $\pi = 2n\varepsilon$;

$$Qq^{2n} + Qq_{2n} = 2n\varepsilon + 2n\varepsilon = 2\pi.$$

i.e. the images q^{2n} and q_{2n} coincide. And if $\frac{\pi}{\varepsilon}$ be an odd number, or $\pi = (2n+1)\varepsilon$,

$$Qq^{2n+1} + Qq_{2n+1} = 4n\varepsilon + 2(\theta + \theta') = (4n + 2)\varepsilon = 2\pi,$$

and the images q^{2n+1} and q_{2n+1} coincide.

If therefore we include the radiant point in the number, the total number of conjugate foci is $\frac{2\pi}{\varepsilon}$.

This theory contains the principle of the kaleidoscope.

II.

Of Images formed by Reflexion at Plane Surfaces.

(43.) From the case of a single pencil we may now proceed to the consideration of an indefinite number of pencils, proceeding from points variously situated.

When a luminous* object is presented to a reflecting surface —inasmuch as the light radiates from each point of the object, in every direction—we may consider the whole light, incident from the object on the mirror, as composed of an indefinite number of pencils of rays, proceeding from each point of the object; and to each such pencil there will be, it is evident, a corresponding pencil and focus of reflected rays. Thus the light incident on the mirror, instead of being reflected to a single focus, will be reflected to an indefinite number of foci, corresponding to the several points of the object; and the aggregate of all these foci of reflected rays composes what is called the *image* of the object.

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* The word *luminous* is taken here, in an enlarged acceptation, to denote any body whatever from which light *radiates*, whether that light be *emitted* directly, or *reflected* from the surface of the body in every direction.

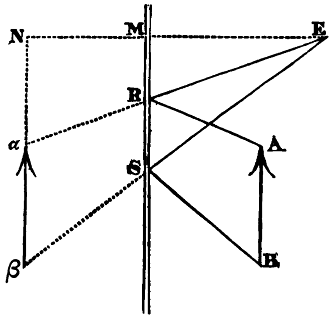
(44.) Now it has been shown, that when a single pencil of rays is incident upon a plane reflecting surface, the foci of the incident and reflected rays are at equal distances from the surface, and at opposite sides. When, therefore, an object is presented to such a surface, its image will be necessarily similar and equal to it in every respect; the several points of the image being similarly situated, with respect to the mirror, as the corresponding points of the object.

It is obvious, however, that as the *faces* of the object and image are opposed, the position of the object with respect to right and left will be inverted in the image.

When the object is inclined to the mirror, the image will be inclined to it also, and at an equal angle. Hence the angle contained by the directions of the object and its image is double the inclination of the object to the mirror; if, therefore, the mirror be inclined to the object at an angle of 45°, the image and object will be at right angles to each other.

(45.) It has been observed that the *foci* of single pencils are distinguished into *real* and *imaginary*, or *virtual*, according as the rays which diverge from, or converge to, these points, actually meet there or not. There is, accordingly, the same distinction among *images*; which are therefore said to be *real*, when the rays proceeding from, or tending to, their several points, actually meet there; *imaginary*, when they meet only when produced. Hence it is evident that the images of objects, formed by reflexion at plane surfaces, are always imaginary; the rays which diverge from their several points never actually intersecting. Such images, however, as far as regards the sensation which they excite in the eye, are as important as any other; for, if the eye be so placed as to receive the pencils diverging from the several points of an image, whether that image be real or imaginary, it will experience the same sensation as if they proceeded from a real object in the same place*.

(46.) If the plane of an object be parallel to that of a mirror, the portion of the mirror which reflects the whole length of the object to the eye, any how placed, is to the length of the object itself, as the perpendicular distance of the eye from the mirror to the sum of the distances of the eye and object from the same.



For, AB being the object presented to the mirror, $\alpha\beta$ its images, and E the place of the eye, if from E the lines Eα, Eβ, be drawn to the extremities of the image, it is evident that the intercepted portion of the mirror, RS, is that which reflects the whole object to the eye. Now, because of similar triangles,

$$RS : \alpha\beta :: ER : E\alpha :: EM : EN,$$

EM being the perpendicular let fall from E upon the mirror, and produced to meet the line of the image in N.

When the eye is at the same distance from the mirror as the object, as in the case of a man viewing his own image, this ratio becomes that of 1 to 2. Hence the portion of a mirror, in which a man may see his entire figure, is half his height.

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* Between real objects and optical images there is this fundamental distinction: the light radiates from the former in every direction, so that they are visible to the eye, any how placed; while from the latter the light issues only in certain directions, so that they are visible only to an eye placed so as to receive the divergent rays.

CHAPTER III.

OF LIGHT REFLECTED AT SPHERICAL SURFACES.

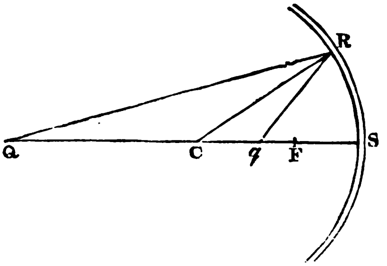
I.

Of the Reflexion of a Pencil of Rays, incident nearly perpendicularly upon a Spherical Surface.

(47.) When a ray of light is incident upon any curved reflecting surface, it will be reflected in the same manner as it would be by the plane which touches the surface at the point of incidence. Hence the investigation of the course of any ray, reflected at such a surface, is reduced to principles already established.

(48.) Given the focus of a small pencil of rays, incident nearly perpendicularly upon a spherical reflecting surface; required the focus of the reflected pencil.

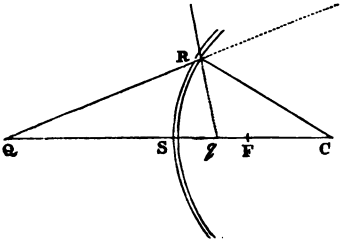
Let RS be any section of the reflecting surface, formed by a plane passing through the centre, and through Q, the focus of the incident pencil; the line QC, joining these points, is perpendicular to the surface at S, and is, therefore, the axis of the pencil. Now, let QR be any ray of the incident pencil, meeting the surface in R, and reflected in the direction Rq; and let RC be the radius drawn to the point of incidence. Then, since this radius is perpendicular to the tangent plane at the point of incidence, the angles, which the incident and reflected rays make with it, are equal. There-



fore, in the triangle QRq, formed by the incident and reflected rays and the intercepted portion of the axis, the vertical angle QRq, or its supplement, is bisected by the radius RC; wherefore (Euclid vi. Prop. 3.), there is $\frac{\text{QR}}{\text{R}q} = \frac{\text{QC}}{\text{C}q}$. But, since the rays composing the pencil are, *quam proximè*, perpendicular to the reflecting surface, and therefore indefinitely near the axis in their incidence, the point R approaches indefinitely to S; and therefore, ultimately,

$$\frac{\text{QS}}{q\text{S}} = \frac{\text{QC}}{q\text{C}},$$

from which we learn that the distances of the foci of the incident and reflected rays from the surface are as their distances from the centre.



(49.) Now, if QS and qS, the distances of the radiant and its conjugate from the surface, be denoted by δ and δ', and the radius CS by r, then, when the reflecting surface is *concave*, QC $= \delta - r$, qC $= r - \delta'$, and there is

$$\frac{\delta}{\delta'} = \frac{\delta - r}{r - \delta'}, \text{ whence } \delta r + \delta' r = 2\delta\delta',$$

or, dividing both sides of the equation by $r \cdot \delta \cdot \delta'$,

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{r},$$

from which it appears that the radius of the surface is an harmonic mean between the distances of the foci from the surface.

(50.) When the reflecting surface is *convex*, denoting the distances from the surface by δ and δ', as before, there is QC $= \delta + r$, qC $= r - \delta'$; whence $\frac{\delta}{\delta'} = \frac{\delta + r}{r - \delta'}$, from which we obtain

$$\frac{1}{\delta'} - \frac{1}{\delta} = \frac{2}{r},$$

an equation differing from the former in the sign of δ, or, which is the same thing, in the signs of δ' and r. Now it appears, from the inspection of the figure, that the distances δ' and r lie, in this case, from the surface in a direction opposite to that in which they were measured in the former; whence it is evident that the formula

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{2}{r}$$

really includes both cases, if we only consider the distances r, δ and δ' as *positive*, when measured from the surface *towards* the incident light; *negative*, when in the opposite direction. This is equivalent to assuming the positive values of r, δ and δ', to belong to the case of a *concave* surface, and *real* foci.

(51.) When the reflecting surface is *plane*, r is infinite, and the formula becomes $\frac{1}{\delta} + \frac{1}{\delta'} = 0$, or

$$\delta + \delta' = 0;$$

from which we learn, as before, that the radiant and its conjugate are at equal distances from the surface, and at opposite sides.

(52.) When the incident rays are *parallel*, δ is infinite, and $\frac{1}{\delta} = 0$; and if we denote the value of δ' in this case by f, the formula becomes $\frac{1}{f} = \frac{2}{r}$, whence

$$f = \frac{r}{2}.$$

This quantity is called the *principal focal distance*, or sometimes simply the *focal length* of the reflector; the focus, into which parallel rays are collected, being called the *principal focus*. In a spherical reflector, therefore, the principal focus bisects the radius.

(53.) Substituting f for $\frac{1}{2}r$, in the general formula (49.), it becomes

$$\frac{1}{\delta} + \frac{1}{\delta'} = \frac{1}{f},$$

from which there is

$$\delta' = \frac{\delta . f}{\delta - f}.$$

Wherefore, the distance of the radiant from the principal focus is to the focal length, as the distance of the radiant from the reflector to the distance of its conjugate from the same.

Again, if we subtract f from δ', there is

$$\delta' - f = \frac{\delta . f}{\delta - f} - f = \frac{f^2}{\delta - f}, \text{ whence}$$

$$(\delta - f)(\delta' - f) = f^2,$$

That is, the distance of the centre from the principal focus is a mean proportional between the distances of the radiant and its conjugate from the same.

(54.) As we shall have frequent occasion, hereafter, to speak of the reciprocals of the distances of the two foci from the surface, it will be convenient to denote them by a characteristic symbol, and to assign them a name. Accordingly, let the quantities $\frac{1}{\delta}$ and $\frac{1}{\delta'}$ be denoted by α and α', respectively; and, since these quantities are the proper measures of the degree of divergence or convergence of the incident and reflected pencils, let them be called, the *vergency of the incident* and *reflected pencil*, respectively.

Again, the reciprocal of the radius is the measure of the *curvature* of the reflecting surface: let it be denoted by the symbol ϱ.

These substitutions being made in the equation (49.), there is

$$\alpha + \alpha' = 2\varrho,$$

which expresses that the sum* of the vergencies of the incident and reflected pencils is a constant quantity.

Let φ denote the value of α', when $\alpha = 0$, or the incident rays are parallel; then there is

$$\varphi = 2\varrho.$$

And substituting in the equation just obtained,

$$\alpha + \alpha' = \varphi;$$

whence it appears that the sum of the vergencies of the incident and reflected pencils is equal to the vergency given to parallel rays by reflexion at the surface.

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* The word *sum* is taken here in its algebraic generality, and becomes the difference when one of the quantities changes sign.

With respect to the signs—since the focal distances are positive, when measured from the surface towards the incident light; negative, when in the opposite direction—it follows that the *positive* values of α, the vergency of the incident rays, denote *divergence*, the *negative*, *convergence*; and that the reverse is the case with respect to α', the vergency of the reflected rays.

(55.) A slight attention to the equation

$$\alpha + \alpha' = 2\varrho$$

will enable us to trace the corresponding values of α and α', the variables which it contains. We shall, in the first place, consider the case in which ϱ is *positive*, or the reflecting surface *concave*.

When $\alpha = 0$, or the incident rays *parallel*, $\alpha' = 2\varrho = \varphi$, and the reflected rays *converge* to the principal focus, or the middle point of the radius.

As the *divergence* of the incident pencil increases, it is evident from the formula that the *convergence* of the reflected pencil diminishes, so that the two foci approach one another; until, when $\alpha = \varrho$, there is also $\alpha' = \varrho$, and the foci, therefore, meet at the centre of the reflecting surface.

When $\alpha > \varrho$, $\alpha' < \varrho$; and as the divergence of the incident pencil is increased continually, the convergence of the reflected pencil is continually diminished; until, when $\alpha = 2\varrho$, $\alpha' = 0$; *i. e.* when the incident rays *diverge* from the principal focus, the reflected rays are *parallel*.

When $\alpha > 2\varrho$, or the divergence of the incident pencil is still further increased, α' becomes *negative*, and the reflected rays *diverge*. And this divergence of the reflected pencil increases continually with that of the incident pencil; until, when α is infinite, α' becomes so too, and the conjugate foci, therefore, again meet at the surface of the reflector.

Finally, when α becomes *negative*, or the incident pencil *convergent*, α' will be always *positive*, or the reflected rays will always *converge*; and the convergence of the reflected pencil will increase continually with that of the incident pencil.

(56.) When the reflecting surface is *convex*, ϱ becomes *negative*, and the formula is

$$\alpha + \alpha' = -2\varrho.$$

Now this change of the sign of ϱ is equivalent to a simultaneous change in the signs of α and α', which, by the rule of signs laid down (54.), amounts to a change of divergent rays into convergent, and *v. v.* Hence all that has been said in the preceding article, respecting the *concave* mirror, is directly applicable to the case of the *convex*, if we substitute *divergence* for *convergence*, and *v. v.*

As the *concave* mirror makes all rays *converge*, except those which diverge from some point between the surface and the principal focus, and of these it diminishes the divergence; so the *convex* mirror will give a *divergence* to all rays, except those which converge to some point within the same limits, whose convergence it will diminish.

(57.) It is sometimes usual to compute the distances *from the centre*, instead of the surface. This is readily done; for, if the distances from the centre, $\mathrm{Q}c$ and qc, be denoted by d and d', then, when the reflecting surface is *convex*, $\mathrm{Q}s = d - r$, $qs = r - d'$, and therefore the relation (48.) is thus expressed,

$$\frac{d}{d'} = \frac{d-r}{r-d'},$$

or, taking away the denominators, dividing the result by rdd', and denoting the reciprocals of r, d and d', by ϱ, u and u', respectively, there is

$$u' + u = 2\varrho.$$

When the reflecting surface is *concave*,

$$\mathrm{Q}s = d + r,\ qs = r - d',\quad \text{whence } \frac{d}{d'} = \frac{d+r}{r-d'},$$

$$\text{whence there is } u' - u = 2\varrho.$$

In this latter case, however, the distances, r and d', are measured, from the centre, in an opposite direction to that in the former case. Wherefore, if the *positive* values of the distances be assumed to belong to the case in which they are measured from the centre, *towards the incident light*, then, in the last-mentioned case, the distances r and d', and therefore their reciprocals ϱ and u' are negative, and the formula

$$u + u' = 2\varrho$$

will include all cases.

If φ denote the value of u', when $u = 0$, or the incident rays parallel; there is

$$\varphi = 2\varrho.$$

And, substituting this in the general equation, we have

$$u + u' = \varphi;$$

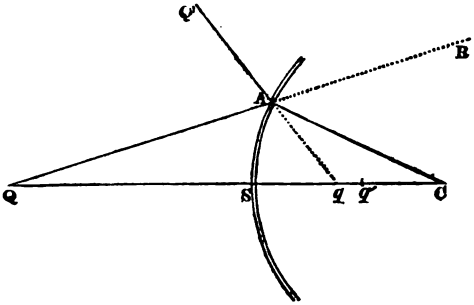
an equation which we shall employ hereafter, when we come to treat of images.

II.

Of Aberration in Reflexion at spherical Surfaces.

(58.) When a pencil of rays, diverging from a point, is reflected at a spherical surface, the intersection of each reflected ray with the axis will, in general, be different; so that the rays composing the reflected pencil do not, as in the case of reflexion at plane surfaces, diverge from, or converge to, a single point. The point which was determined in the last section, and which is called the *geometric focus* of the reflected rays, is the limit of the intersections, with the axis, of rays, which approach it indefinitely, and are therefore, *quam proximè*, perpendicular in their incidence. The importance of this point in all optical applications is derived from this—that, although no ray, however slightly removed from the axis in its incidence, shall, after reflexion, meet it exactly in that point, yet, whatever be the breadth of the pencil, the number of rays collected in a given indefinitely small space, is infinitely greater at this point than at any other, as will appear more fully when we come to treat of caustics.

(59.) Let us now proceed to inquire, generally, the intersection of any reflected ray whatever with the axis.



AS being the section of the reflecting surface*, QC the axis of the incident pencil, QA any incident ray which is reflected in the direction AQ', and AC the radius of the reflecting surface drawn to the point of incidence; in the triangles QAC, qAC, we have the relations

$$\frac{\text{AC}}{\text{CQ}} = \frac{\sin.\,\text{CQA}}{\sin.\,\text{CAQ}}, \quad \frac{\text{AC}}{\text{C}q} = \frac{\sin.\,\text{C}q\text{A}}{\sin.\,\text{CA}q},$$

or, denoting CQ and Cq, the distances of the intersections of the incident and reflected rays with the axis from the centre, by d and d'; the angle at the centre, ACS, by θ; and CAB = CAq, by ι,

$$\frac{r}{d} = \frac{\sin.(\iota - \theta)}{\sin.\,\iota}, \quad \frac{r}{d'} = \frac{\sin.(\iota + \theta)}{\sin.\,\iota},$$

and, adding,

$$\frac{r}{d'} + \frac{r}{d} = \frac{\sin.(\iota + \theta) + \sin.(\iota - \theta)}{\sin.\,\iota} = 2\cos.\,\theta.$$

Whence—dividing by r, and denoting the reciprocals of r, d and d', by ϱ, u and u', as before—there is

$$u' + u = 2\varrho\,.\cos.\,\theta\,;$$

an equation which determines the intersection of the reflected ray with the axis, whatever be its incidence.

(60.) When the incident ray is parallel to the axis, $u = 0$; and, if φ denote the value of u' in this case, there is

$$\varphi = 2\varrho\,.\cos.\,\theta, \quad \text{or } f = \tfrac{1}{2}r\,.\sec.\,\theta.$$

Whence, if a tangent be drawn to the reflecting curve at the point of incidence, the reflected ray will bisect the portion of the axis intercepted between it and the centre.

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* The annexed figure belongs to the case of *divergent* rays, incident upon a *convex* surface; in which the distances d, d' and r, are all *positive*. The resulting equation, however, includes all cases, if the rule respecting the signs (57.) be attended to.

If, for $2\varrho \, . \cos. \theta$, its value, φ, be substituted, in the equation of the preceding article, there is

$$u' + u = \varphi;$$

an equation remarkable for its simplicity, and which is precisely analogous to that found (57.), the quantity denoted by φ, however, being, in this case, variable with the aperture.

(61.) Now, when a pencil of rays, diverging from a point, is incident upon the reflecting surface, it is evident from the equation,

$$u' + u = 2\varrho \, . \cos. \theta,$$

that the value of u' is, in general*, different for each ray of the incident pencil. If $u_{\prime}$ denote its *ultimate* value, when $\theta' = 0$, there is

$$u_{\prime} + u = 2\varrho,$$

agreeing with the formula found (57.).

Now, if we subtract the latter of these equations from the former, there is $u' - u_{\prime} = 2\varrho \, (\cos. \theta - 1)$. But $\cos. \theta - 1 = -$ ver. sin. θ; wherefore, if the versed sine be denoted by v, and the difference between u' and its ultimate value by $\Delta u'$, this result is thus expressed:

$$\Delta \, . \, u' = - 2\varrho \, . \, v.$$

(62.) q' being the *geometric focus*, or the focus of those rays which are indefinitely near the axis in their incidence, the distance qq', between it and the point in which the extreme ray cuts the axis, is called the *aberration* of the extreme ray.

This quantity is readily found; for, if $d_{\prime}$ denote the *ultimate* value of d', we have

$$\Delta u' = u' - u_{\prime} = \frac{1}{d'} - \frac{1}{d_{\prime}}; \text{ whence there is}$$

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* There are two particular cases, in which u' is invariable, whatever be the value of θ, or all the rays of the pencil reflected accurately to the same point. These are, 1st, when $\varrho = 0$, or the reflecting surface plane; and 2dly, when u is infinite, or the radiant at the centre; in which case u' is also infinite, and the reflected rays all meet in the same point.

$$d' = d_{\prime}[1 + d_{\prime} . \Delta u]^{-1} = d_{\prime}[1 - \frac{d_{\prime}}{f} v]^{-1};$$

and, if this be expanded by the binomial theorem,

$$d' = d_{\prime}[1 + \frac{d_{\prime}}{f} v + \left(\frac{d_{\prime}}{f} v\right)^2 + \left(\frac{d_{\prime}}{f} v\right)^3 + \&c.]$$

When the incident rays are parallel, $d' = f$, $d_{\prime} = f_{\prime}$, and this series becomes

$$f = f_{\prime}[1 + v + v^2 + v^3 + \&c.];$$

a series which might have been obtained directly from the general value of f, namely $\frac{1}{2} r. \sec. \theta$.

The general value of the aberration, $d' - d_{\prime}$, is given by the preceding series, being equal to that series wanting the first term. When the angle θ, however, is small, the square and higher powers of its versed sine may be neglected, and the approximate value of the aberration, in this case, becomes

$$d' - d_{\prime} = \frac{d_{\prime}^2}{f} . v.$$

When the incident rays are parallel, $d_{\prime} = f$, and its value becomes simply

$$f . v.$$

(63.) These values of the aberration are generally expressed in terms of the aperture of the reflecting surface. Let x denote the semiaperture, or the sine of θ to the radius r, then

$$v = \tfrac{1}{2} . \sin.^2 \theta, \text{ nearly, } = \tfrac{1}{2} . \frac{x^2}{r^2}; \text{ also } f = \tfrac{1}{2} r.$$

Wherefore, substituting, the general expression of the aberration becomes

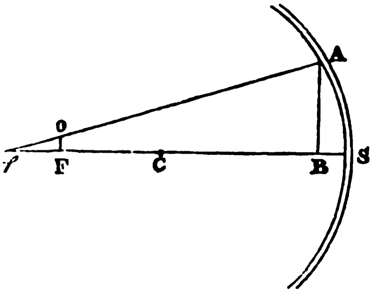
$$d' - d_{\prime} = \frac{d_{\prime}^2 x^2}{r^3};$$

and for parallel rays, $d_{\prime} = f = \frac{1}{2} r$, and it becomes

$$\frac{x^2}{4r}.$$

(64.) There is another species of aberration sometimes taken into account; this is the perpendicular to the axis, erected at the geometric focus, and terminated by the extreme ray. This is called the *lateral aberration*, in contradistinction to the former, which is termed the *longitudinal* aberration.

Thus, if SF be the axis of the pencil, F the geometric focus of the reflected rays, and Af the extreme reflected ray, meeting the axis in f, and the perpendicular FO in O; then Ff is the longitudinal, and FO the lateral aberration, and we have



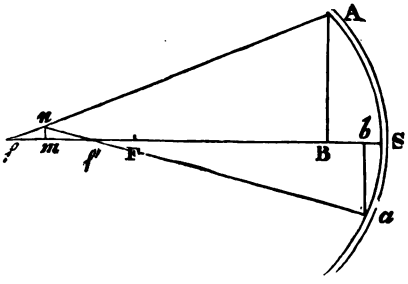
$$\text{FO} = \text{F}f \cdot \frac{\text{AB}}{\text{B}f} = \text{F}f \cdot \frac{\text{AB}}{\text{SF}}, \text{ very nearly.}$$

Wherefore, if the *longitudinal aberration*, Ff, be denoted by $\varkappa \cdot x^2$,

$$lat.\ aberration = \frac{\varkappa}{\delta} \cdot x^3.$$

(65.) It is of importance to determine the least possible space into which all the reflected rays are collected; for this, being the *physical* focus of the radiant point, furnishes a measure of the confusion arising from the aberration.

To find this, let Af be the extreme ray, cutting the axis in f; af' any reflected ray at the other side of the axis, and meeting it in f'; and mn the perpendicular distance of the intersection of these rays from the axis. Then it is evident that, as the arch Sa increases, this perpendicular shall first increase, and afterwards decrease—increasing, on account of the increase of the angle af'S, and then decreasing, on account of the decrease of ff',—until, when Sa = SA, f' coincides with f, and mn vanishes altogether. mn has, therefore, a *maximum* value, and, when it reaches this value, it is evident that all the rays at the same side of the axis with af' shall pass through it, and that it is the least possible space into which



those rays can be collected. *mn* is, therefore, in this case, the radius of the *least circle of aberration.*

To find its magnitude—let the semiapertures SA and *sa* be denoted by x and x'; and let $\varkappa$ denote, as before, the coefficient of the square of the aperture in the value of the aberration; then, F being the geometric focus, we have

$$\mathrm{F}f = \varkappa \,.\, x^2, \quad \mathrm{F}f' = \varkappa \,.\, x'^2, \text{ whence, subtracting,}$$

$$ff' = \varkappa\,(x^2 - x'^2).$$

But, on account of the similar triangles *mnf*, ASf — *mnf′*, *asf′*, there is

$$nf = mn \,.\, \frac{\mathrm{S}f}{\mathrm{AS}} = \frac{\varrho \,.\, \delta}{x} \,.\, q.p. \quad nf' = mn \,.\, \frac{\mathrm{S}f'}{as} = \frac{\varrho \,.\, \delta}{x'} \,.\, q.p.$$

mn being denoted by ϱ. Whence, adding, there is

$$ff' = \varrho \,.\, \delta\left(\frac{1}{x} + \frac{1}{x'}\right);$$

and finally, equating these two values of ff', we obtain

$$\varrho = \frac{\varkappa}{\delta}\,(x - x')\,xx'.$$

Now, x' being the variable in this expression, ϱ varies as $(x - x')x'$, and is a maximum when the latter is so. But the product $(x - x')x'$ is evidently a maximum when $x' = \frac{1}{2}x$; and becomes, in that case, $\frac{1}{4}x^2$. Wherefore, the maximum value of ϱ, or the radius of the least circle of aberration, is

$$\varrho = \tfrac{1}{4}\,\frac{\varkappa}{\delta} \,.\, x^3;$$

from which we learn that the radius of the least circle of aberration is one-fourth of the lateral aberration of the extreme ray.

(66). The general value of $\varkappa$ (63.) is $\dfrac{d^2}{r^3} = \dfrac{(\delta - r)^2}{r^3}$, in which δ is the distance of the focus of reflected rays from the surface. Wherefore, substituting,

$$\varrho = \tfrac{1}{4}\,\frac{(\delta - r)^2}{\delta}\;\frac{x^3}{r^3}.$$

In the case of parallel rays, $\delta = \frac{1}{2}r$, and this becomes

$$\varrho = \tfrac{1}{8}\,\frac{x^3}{r^2}, \quad \text{or,} \quad 2\varrho = \frac{x^3}{(2r)^2};$$

from which we learn that the diameter of the least circle of aberration, for parallel rays, is equal to the cube of the semi-aperture divided by the square of the diameter of the reflecting surface.

(67.) With respect to the position of the centre of this circle, we have $nf = \frac{\rho \cdot \delta}{x} = \frac{1}{4}\varkappa . x^2$; but $\mathrm{F}f = \varkappa . x^2$; wherefore, subtracting, there is

$$n\mathrm{F} = \tfrac{3}{4}\varkappa . x^2;$$

or, the distance of the centre of the least circle of aberration from the geometric focus is equal to three-fourths of the longitudinal aberration of the extreme ray.

For parallel rays, therefore, $n\mathrm{F} = \frac{3}{16}\frac{x^2}{r}$.

III.

Of Images formed by Reflexion at spherical Surfaces.

(68.) When an object is presented to a spherical reflecting surface, its image is the aggregate of the foci of the reflected pencils, corresponding to the several points of the object. Hence, to obtain the form and position of the image, we must determine the focus conjugate to each point of the object. This is done by drawing, from each point of the object, a line through the centre of the spherical surface, and computing, on this line, which is the axis of the pencil, the place of the focus of the reflected pencil, by means of the formula for conjugate foci (57.).

Accordingly, the *relative position* of the object and image is determined by the same rules, as that of the conjugate foci. The object and its image, therefore, lie always on the same side of the principal focus; they move in opposite directions, and meet at the centre and surface of the reflector.

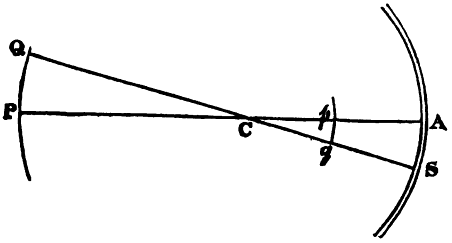
As the axes of the several pencils intersect at the centre of the spherical surface, it is evident that the image will be *inverted* with respect to the object, when they lie at opposite sides of the centre; *erect*, when at the same side. From this it follows that the image of an object presented to a convex mirror is always erect; while that produced by a concave mirror is erect only when the object is between the principal focus and the surface, and in all other cases inverted.

It is also evident that the image is *real* when inverted, and when erect *imaginary*.

(69.) If the surface of the object which is presented to the reflector be a regular surface, there must subsist some relation between the distances of the several points of that surface from the centre of the mirror, and their inclinations to some fixed line drawn through the same point. Wherefore, combining this relation with that of the distances themselves, given by the formula

$$u' + u = \varphi,$$

we shall have the relation between the distances of the several points of the image from the centre, and the angles which they form with the given line; that is, we shall have the polar equation of the curve which is the section of the image.



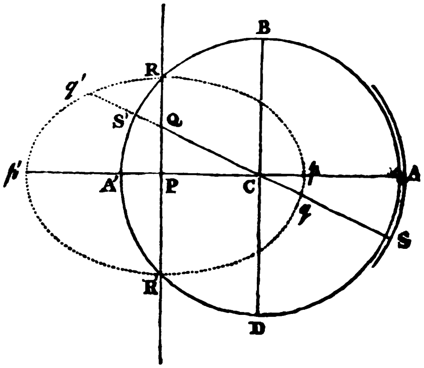
Thus, when the figure of the object is a spherical surface concentric with the reflector, whose section is represented by PQ, u, the reciprocal of the radius, CQ, is constant, and therefore $u' = \varphi - u$, is also constant, and the figure of the image, pq, is likewise a spherical surface concentric with the reflector, whose radius, Cq, $= \frac{1}{u'}$.

If, for example, the object presented to a concave reflector be a portion of its own sphere, $u = \varrho$; and since $\varphi = -2\varrho$, there is $u' = -3\varrho$. Or, the radius of the spherical image will be one-third of that of the reflector.

(70.) If the surface of the object be plane, that of its image will be that generated by the revolution of a conic section round its axis.

Let us take, as the axis, the diameter of the spherical reflector which is perpendicular to the plane of the object; and consider the sections of the reflector, the object and image, formed by any plane passing through this line. Then PQ being the rectilinear section of the object, QC the line drawn from any point of it to the centre, there is CP = CQ . cos. PCQ; that is, if the reciprocals of the distances CP and CQ be denoted by a and u respectively, and the angle PCQ by θ,

$$u = a \cos. \theta.$$



Wherefore, substituting this value of u, in the equation $u' = \varphi - u$,

$$u' = \varphi - a . \cos. \theta,$$

the equation of a conic section, whose *focus* is the centre of the spherical surface, and *axis major* the diameter perpendicular to the plane of the object.

Comparing this with the known polar equation of the conic section, in which the angles are measured from the *remote vertex*,

$$\frac{1}{\varrho} = \frac{1}{p} - \frac{\varepsilon}{p} . \cos. \omega,$$

in which p denotes the *semiparameter*, and ε the *excentricity*; we have $\varphi = \frac{1}{p}$, and $a = \frac{\varepsilon}{p}$; or, if f be the focal length, and d the distance of the object from the centre,

$$p = f, \quad \text{and } \epsilon = \frac{f}{d}.$$

Hence the *parameter* of the section is equal to $2f$, or to r, the radius of the spherical surface, and therefore constant, whatever be the position of the object. Now, the principal parameter is equal to the diameter of curvature at the vertex of the conic section; and, accordingly, we derive the remarkable conclusion, that, whatever be the position of the object, and consequent magnitude of the section, the curvature at the vertex is invariable.

(71.) With respect to the nature of the section, it will be an *ellipse*, *hyperbola*, or *parabola*, according as the excentricity ϵ is less, greater than, or equal to, unity; that is, according as d, the distance of the object from the centre, is greater, less than, or equal to, f, the principal focal length.

We have now all the conditions requisite to assign the nature of the section for each position of the object.

When the object is beyond the centre at an infinite distance, then $\epsilon = \frac{f}{d}$ is nothing; and the image is a *circle* whose radius is f, half the radius of reflector.

As the object approaches the centre from an infinite distance, the image becomes an *ellipse* of increasing excentricity. When the object reaches the middle point of the radius, the excentricity ϵ becomes unit, and the ellipse is changed into a *parabola*.

When the distance from the centre becomes less than half radius, ϵ is greater than unity, and the curve becomes an *hyperbola*, of increasing excentricity as the object approaches the centre; and when the object reaches the centre, the excentricity is infinite, and the hyperbola becomes a *straight line* coincident with the object.

When the object passes the centre, and moves from thence to an infinite distance, the same changes take place at the corresponding distances, as in the approach to the centre from an infinite distance—the curve being first an hyperbola, then a parabola, then an ellipse, and lastly a circle.

In this case, however, a becomes negative, and the equation is

$$u' = \varphi + a\,.\cos.\,\theta,$$

the equation of a conic section, in which the angles are measured from the *near vertex;* and, accordingly, the axis of the section lies in an opposite direction, with respect to the centre, to that in the former case.

(72). We have here considered the question in all its mathematical generality—considering the line of the object to extend *indefinitely* in both directions—and the reflector to be an *entire* circle.

When the object is entirely without the circle, it will be evident, from the inspection of the figure, that if a right line be drawn through the centre perpendicular to the axis, the concave hemisphere will give by reflexion the portion of the section between the near vertex and focal ordinate, while the convex hemisphere will give the remainder.

When the rectilinear object intersects the circle, the remote hemisphere will give, as before, the portion between the near vertex and perpendicular diameter; the portion of the circle intercepted between the object and perpendicular diameter will give the portion of the image included by the same lines, by reflexion at the convex surface; and the lesser segment of the circle will reflect the remainder of the image at its concave surface.

When the right line is limited, or the reflector but a portion of a circle, the curvilinear image will be only a portion of a conic section, terminated by lines drawn from the centre, either to the extreme points of the object or extreme points of reflector.

It would be useless, as well as troublesome, to consider the figure of the image corresponding to any other figure of the object, inasmuch as the objects presented to reflecting surfaces, if not plane, are generally of irregular forms; the images of which are therefore only to be determined by considering separately each point of the object.

(73.) We shall now consider the *magnitude* of the images produced by reflexion at a spherical surface.

If the section of the object be considered as an arc concentric with the reflector (a supposition which will not differ much from the truth when the object is small and perpendicular to the axis), we have seen that the image will be likewise an arc having the same centre; and since these arcs, subtending the same angle at the centre, are to one another as their radii, see fig. page 44, it follows that the linear magnitudes of the object and image are to one another as their distances from the centre of the reflecting surface; wherefore, denoting these magnitudes by m and m', and observing that the distances of the conjugate foci from the centre are to one another as their distances from the surface, there is

$$\frac{m'}{m} = \frac{d'}{d} = \frac{\delta'}{\delta};$$

$$\text{or, since } \delta' = \frac{\delta . f}{\delta - f}, \quad \frac{m'}{m} = \frac{f}{\delta - f}.$$

When the object is infinitely distant, δ is infinite, and $\frac{m'}{m} = 0$; that is, the image is infinitely small compared with the object.

As δ diminishes, $\frac{m'}{m}$ increases; until, when $\delta = f$, $\frac{m'}{m}$ becomes infinite; that is, when the object is at the principal focus, the image is infinitely great compared with it.

As δ is still further diminished, the ratio $\frac{m'}{m}$ also diminishes, until when $\delta = 0$, $\frac{m'}{m} = 1$; or, when the object comes to the surface, the image is equal to it.

When the object moves to the *convex* side of the surface, the focal length f is *negative*, and the formula becomes $\frac{m'}{m} = \frac{-f}{\delta + f}$; from which it appears that, in this case, the ratio is always less than unity, or the image always smaller than the object; the ratio decreasing indefinitely as the distance of the object increases.

CHAPTER IV.

OF LIGHT REFLECTED AT ANY CURVED SURFACES.

I.

General Theory of Reflexion at any curved Surfaces.

(74.) In what follows we shall limit our attention to the case in which the reflecting surface is a surface of revolution, and the incident ray in a plane passing through its axis; a case which comprehends most of those with which we are concerned in the practical applications of optical science.

As the normal to the surface, in surfaces of revolution, lies always in a plane passing through the axis, the plane of incidence, in this case, and therefore also that of reflexion, must be a plane containing the axis of revolution: we may, therefore, in what follows, confine our attention to the section of the surface made by such a plane, and containing the incident ray.

The angles, which the incident and reflected rays make with the axis of abscissæ, being denoted by ω and ω', the cosines of the angles, which the incident ray makes with the axes of y and x, are, sin. ω, cos. ω; those of the angles contained by the reflected ray with the same, sin. ω', cos. ω', respectively. But the cosines of the angles, which the tangent to the curve at the point of incidence makes with the same axes, are $\frac{dy}{ds}$, $\frac{dx}{ds}$; therefore the cosines of the angles which the incident and reflected rays make with the tangent to the curve at the point of incidence are, respectively,

$$\cos.\omega.\frac{dx}{ds}+\sin.\omega.\frac{dy}{ds},$$
$$\cos.\omega'.\frac{dx}{ds}+\sin.\omega'.\frac{dy}{ds}.$$

Now these angles are $\frac{\pi}{2}$ − angle of incidence, and $\frac{\pi}{2}$ + angle of reflexion, respectively, and therefore (since the angles of incidence and reflexion are equal) their cosines are equal with opposite signs; that is,

$$\cos.\omega.dx+\sin.\omega.dy+\cos.\omega'.dx+\sin.\omega'.dy=0;$$

an equation which determines ω', the angle made by the reflected ray with the axis of abscissæ, as a function of ω, the angle made by the incident ray with the same, and the coordinates of the point of incidence.

(75.) Let (α, β) be the coordinates of any point of the incident ray, (α', β') those of the reflected ray; then, as these rays pass through the point of incidence, whose coordinates are (x, y), their equations are

$$\beta-y=\tan.\omega.(\alpha-x),$$
$$\beta'-y=\tan.\omega'(\alpha'-x);$$

and if we eliminate ω and ω' between these equations and that of the preceding article, the result will give a relation between (α, β), (α', β'), and the coordinates of the reflecting curve. Accordingly, when the coordinates (α, β) are determined by the law to which the incident light is subject, the resulting equation gives the relation between α' and β', or the equation of the reflected ray, for each point of the reflecting curve.

To proceed with this elimination:—From the equations of the incident and reflected ray we deduce

$$\cos.\omega=\frac{\alpha-x}{\rho},\quad \sin.\omega=\frac{\beta-y}{\rho},$$
$$\cos.\omega'=\frac{\alpha'-x}{\rho'},\quad \sin.\omega'=\frac{\beta'-y}{\rho'},$$

making, for abbreviation,

$$\sqrt{(\alpha-x)^2+(\beta-y)^2}=\rho,$$
$$\sqrt{(\alpha'-x)^2+(\beta'-y)^2}=\rho';$$

and substituting these values in the preceding equation, it becomes

$$\frac{(\alpha - x)dx + (\beta - y)dy}{\varrho} + \frac{(\alpha' - x)dx + (\beta' - y)dy}{\varrho'} = 0;$$

which, when α and β are given, furnishes the relation between α' and β', or the equation of the reflected ray.

When the incident ray is parallel to the axis, $\beta - y = 0$, wherefore $\varrho = \alpha - x$, and the preceding equation is reduced to

$$\varrho' . dx + (\alpha' - x)dx + (\beta' - y)dy = 0.$$

(76.) If we differentiate the values of ϱ and ϱ', relatively to x and y only, we find

$$(\alpha - x)dx + (\beta - y)dy = -\varrho . d\varrho$$
$$(\alpha' - x)dx + (\beta' - y)dy = -\varrho' . d\varrho';$$

which being substituted in equation just obtained, it becomes

$$d\varrho + d\varrho' = 0;$$

a remarkable result, from which it follows that the quantity $(\varrho + \varrho')$ is a *minimum*, or that the course of the light from any assumed point in the incident ray, to any assumed point in the reflected ray, is the shortest possible.

When the incident ray is parallel to the axis, $d\varrho = -dx$, and this equation becomes

$$d\varrho' = dx.$$

(77.) To determine the curve which will reflect rays, proceeding from a point, *accurately* to a point, we have only to consider $(\alpha\,\beta)$, $(\alpha'\,\beta')$, as given points in equat. (75.), and integrate on that supposition. For the sake of simplification we shall take the focus of the reflected rays as the origin of the coordinates, or make $\alpha' = 0$, $\beta' = 0$; then integrating, we find

$$\sqrt{(\alpha - x)^2 + (\beta - y)^2} + \sqrt{x^2 + y^2} = a.$$

If this be transformed to polar coordinates, by making

$$x = r . \cos. \omega, \; y = r . \sin. \omega, \; x^2 + y^2 = r^2,$$

there is

$$(\alpha - r\cos. \omega)^2 + (\beta - r\sin. \omega)^2 = (a - r)^2;$$

and developing, we find

$$2r = \frac{a^2 - (\alpha^2 + \beta^2)}{a - (\alpha \cos. \omega + \beta \sin. \omega)};$$

or, making $\alpha = a\varepsilon . \cos. \gamma$, $\beta = a\varepsilon . \sin. \gamma$,

$$r = \frac{\frac{1}{2}a(1 - \varepsilon^2)}{1 - \varepsilon . \cos. (\omega - \gamma)}.$$

The equation of an *ellipse* or *hyperbola*, whose excentricity is ε, axis major a, and distance between the foci, $a\varepsilon = \sqrt{\alpha^2 + \beta^2}$.

Also, the axis major coincides with the line joining the foci of the incident and reflected rays, since it makes the angle γ with the axis of abscissæ. Hence it appears that, as the focus of the reflected rays is one of the foci of the curve, namely the origin, so the focus of incident rays is the other.

When the incident rays are parallel, we must return to the differential equation (75.), in which making $\alpha' = 0$, $\beta' = 0$, as before, and integrating, there is

$$x + \sqrt{x^2 + y^2} = \text{const.}$$

If x' be the value of x, when $y = 0$, then const. $= 2x'$. Substituting this value, and transforming to polar coords.,

$$r = \frac{2x'}{1 + \cos. \omega},$$

the equation of a *parabola*, whose principal parameter is equal to $4x'$.

These physical properties of the conic sections readily follow from the geometric properties of those curves. For, in the ellipse and hyperbola, since the lines, drawn from any point to the foci, make equal angles with the tangent at that point, it follows that rays diverging from, or converging to, one focus of an ellipse, will, after reflexion, converge to, or diverge from, the other; and that rays diverging from, or converging to, one focus of an hyperbola, will diverge from, or converge to, the other. Again, in the parabola, since the lines drawn from any point of the curve—one to the focus, and the other parallel to the axis—make equal angles with the tangent at that point; it follows that rays incident upon a parabola, in a direction parallel to the axis, will, after reflexion, converge to, or diverge from the focus, according as they are incident on the concave or convex surface.

But though it is thus evident from geometric considerations that the conic sections possess this property, yet it remains to show that they are the *only curves* possessing it: and this the analytic demonstration, above given, has done.

(78.) The equation of the reflected ray (75.) may be put under a more convenient form. For, if we make $dy = pdx$, and multiply by $\varrho\varrho'$, it becomes

$$\varrho'\left[(\alpha - x) + p(\beta - y)\right] + \varrho\left[(\alpha' - x) + p(\beta' - y)\right] = 0.$$

Now, transferring the 2d member, squaring both sides, and substituting for ϱ^2 and ϱ'^2 their values, the resulting equation, it will be found, is divisible by the factor,

$$(\beta - y)(\alpha' - x) - (\beta' - y)(\alpha - x),$$

and is thus reduced to

$$(p^2 - 1)\left[(\beta - y)(\alpha' - x) + (\beta' - y)(\alpha - x)\right]$$
$$+ 2p\left[(\alpha - x)(\alpha' - x) - (\beta - y)(\beta' - y)\right] = 0,$$

which is equivalent to

$$(\beta' - y)\left[(p^2 - 1)(\alpha - x) - 2p(\beta - y)\right]$$
$$+ (\alpha' - x)\left[(p^2 - 1)(\beta - y) + 2p(\alpha - x)\right] = 0$$[*].

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[*] This equation of the reflected ray may readily be obtained directly. ω and ω' denoting, as before, the angles which the incident and reflected rays make with the axis of abscissæ, let θ denote the angle made by the normal to the curve at the point of incidence with the same; then, the angle of incidence $= \pm (\omega - \theta)$; the angle of reflexion $= \pm (\theta - \omega')$. And, since these angles are equal, we have

$$\omega - \theta + \omega' - \theta = 0, \text{ or } \omega + \omega' = 2\theta.$$

Hence $\tan.(\omega + \omega') = \tan. 2\theta$, or $\dfrac{\tan.\omega + \tan.\omega'}{1 - \tan.\omega\,.\tan.\omega'} = \dfrac{2\tan.\theta}{1 - \tan.^2\theta}$;

but $\tan.\theta = -\dfrac{dx}{dy} = -\dfrac{1}{p}$, whence $\dfrac{2\tan.\theta}{1 - \tan.^2\theta} = -\dfrac{2p}{p^2 - 1}$;

and therefore

$$(p^2 - 1)(\tan.\omega + \tan.\omega') + 2p(1 - \tan.\omega\,.\tan.\omega') = 0.$$

And substituting for $\tan.\omega$ and $\tan.\omega'$ their values, $\dfrac{\beta - y}{\alpha - x}$, $\dfrac{\beta' - y}{\alpha' - x}$ (75.), we obtain the equation above written.

When the incident ray is parallel to the axis, $\beta - y = 0$; and the resulting equation is divisible by $\alpha - x$. Wherefore, omitting the *traits*, which are no longer necessary, the equation of the reflected ray, in this case, becomes

$$(p^2 - 1)(\beta - y) + 2p(\alpha - x) = 0.$$

When the incident rays diverge from a point in the axis, $\beta = 0$; and if we place the origin, which is arbitrary, at this point, we have also $\alpha = 0$, and the general equation of the reflected ray becomes

$$(\beta - y)\left[(p^2 - 1)x - 2py\right] + (\alpha - x)\left[(p^2 - 1)y + 2px\right] = 0.$$

(79.) To get the point in which the reflected ray meets the axis, we have only to make $\beta = 0$ in its equation, and the resulting value of α is the coordinate of the point required.

Thus, the last found equation may be put under the form

$$(\beta - y)\left[p(px - y) - (x + py)\right] + (\alpha - x)\left[p(x + py) + (px - y)\right] = 0;$$

from which, making $\beta = 0$, we obtain

$$\alpha\left[p(x + py) + (px - y)\right] = 2(x + py)(px - y),$$

which is equivalent to the following:

$$\frac{2}{\alpha} = \frac{p}{px - y} + \frac{1}{x + py}.$$

To get the *geometric* focus, or the focus of rays indefinitely near the axis in their incidence, we should substitute for p its value given by the equation of the reflecting curve, and make $y = 0$ in the result. It is easy to see how these results may be applied to the investigation of the *aberration*, &c.

(80.) Thus, let the reflecting surface be a *right cone*, and the incident rays diverge from a point in the axis; then, δ denoting the distance of the vertex of the cone from the radiant point, or origin, and θ its semiangle, the equation of the generating right line will be

$$y = \tan.\theta(x - \delta), \quad \text{whence } p = \tan.\theta.$$

And substituting these values of y and p, in the equation of the preceding article, we obtain

$$\frac{2}{\alpha} = \frac{1}{\delta} + \frac{1}{x + \tan.^2\theta\,(x - \delta)}.$$

Or, if we remove the origin to the vertex of the cone, by substituting $\alpha + \delta$ for α, $x + \delta$ for x, we find

$$\alpha = \frac{\delta \,.\, x}{x + 2\delta \,.\, \cos.^2\theta}, \quad \text{or} \quad \frac{1}{\alpha} = \frac{1}{\delta} + \frac{2\cos.^2\theta}{x}.$$

When the incident rays are parallel, or the radiant point infinitely distant, $\frac{1}{\delta} = 0$; and there is

$$\alpha = \frac{x}{2\cos.^2\theta}.$$

When $\theta = 90^\circ$, the conical surface becomes a plane perpendicular to the axis. In this case, $\cos.\,\theta = 0$; and, for the intersection of the reflected ray with the axis, we have

$$\alpha = \delta,$$

showing that the points, in which the incident and reflected rays meet the axis, are equally distant from the reflecting surface, and at opposite sides. Further, as this value of α is constant, it follows that all rays, diverging from the same point, and incident on a plane surface, will, after reflexion, diverge accurately from the same point, agreeably to that which has been already established (39.).

(81.) To apply the preceding theory to the case of the *sphere:* let δ be the distance of the radiant point from the centre; then, if we transfer the origin from the former to the latter point, by substituting $\alpha + \delta$ for α, $x + \delta$ for x, the equation of the reflected ray will be

$$(\beta - y)\left[(p^2 - 1)(x + \delta) - 2py\right]$$
$$+ (\alpha - x)\left[(p^2 - 1)y + 2p(x + \delta)\right] = 0.$$

Now, the equation of the circle, referred to the centre, is

$$y^2 + x^2 = r^2, \text{ whence } ydy + xdx = 0, \text{ and } p = -\frac{x}{y}.$$

Substituting this value of p, the preceding equation becomes

$$(\beta - y)\left[\delta(x^2 - y^2) + r^2x\right] = (\alpha - x)\left[2\delta xy + r^2y\right],$$

which is equivalent to

$$\beta\left[\delta(x^2 - y^2) + r^2x\right] - \alpha\left[2\delta xy + r^2y\right] + r^2\delta y = 0.$$

This equation may be conveniently transformed, by expressing x and y in terms of the angle at the centre; for if this angle be denoted by ω, there is

$$x = r.\cos.\omega, \quad y = r.\sin.\omega;$$

wherefore, substituting these values, and observing that

$$\cos.^2\omega - \sin.^2\omega = \cos.2\omega, \quad 2\cos.\omega.\sin.\omega = \sin.2\omega,$$

the equation of the reflected ray becomes

$$\beta[\delta.\cos.2\omega + r.\cos.\omega] - \alpha[\delta.\sin.2\omega + r.\sin.\omega] + \delta r.\sin.\omega = 0.$$

(82.) To find the point in which the reflected ray meets the axis, let $\beta = 0$ in the last equation, and we find

$$\alpha[\delta.\sin.2\omega + r.\sin.\omega] = \delta r.\sin.\omega;$$

or, dividing both sides by $\alpha\delta r.\sin.\omega$,

$$\frac{1}{\alpha} = \frac{1}{\delta} + \frac{2\cos.\omega}{r};$$

agreeing with the equation already obtained (59.), and from which the whole theory of focal lengths, aberrations, &c. is deduced.

(83.) The equation of the reflected ray, just found, might also be employed for the determination of ω, when α and β are given; or, in other words, for the determination of the point of incidence, the incident and reflected rays passing each through a given point. This is the celebrated problem of Alhazen, and is evidently equivalent to the following geometrical problem: "Given the two foci, to describe an ellipse which shall touch a given circle." The algebraic solution, by means of the above equation, leads to an equation of the 4th dimension. There is one case, however, in which the solution is easy; it is that in which the right line joining the given points passes through the centre; in this case $\beta = 0$, and the equation is reduced to that of the preceding article, from which we get

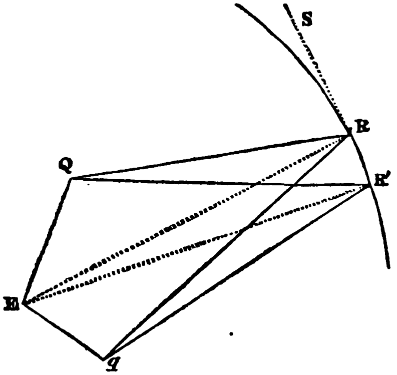
$$\cos.\omega = \frac{r}{2}\left(\frac{1}{\alpha} - \frac{1}{\delta}\right).$$

II.

Of Caustics produced by Reflexion at any curved Surface.

(84.) When a pencil of rays is reflected at any surface, the successive reflected rays will, in general, intersect in different points. Now, the *locus* of these points, or the curve traced out by the successive intersections of the reflected rays, is called the *caustic;* and it is evident that the condensation of the light in this curve is infinitely greater than at any other points. The investigation of these curves is, accordingly, of considerable importance in optical researches.

Q being the focus of the incident rays, let QR, QR$'$ be two successive incident rays, Rq, R$'q$, the corresponding reflected rays, meeting in q; then the caustic curve is the locus of the point q. Let RE, R$'$E, be two normals to the reflecting curve, erected at R and R$'$, the successive points of incidence, and therefore meeting at the centre of the osculating circle. Then, if we denote QE and qE, the distances of the intersections of the incident and reflected rays from that centre, by D and d; QR and Rq, the distances of the same points from the point of incidence on the reflecting curve, by ρ and ρ'; RE, the radius of the osculating circle, by r; and QRS, the angle which the incident ray makes with the tangent to the curve at the point of incidence, by θ, there is



$$D^2 = r^2 + \rho^2 - 2r\rho \, . \sin. \theta,$$
$$d^2 = r^2 + \rho'^2 - 2r\rho' \, . \sin. \theta.$$

But since, in proceeding from the point R in the reflecting curve, to the consecutive and indefinitely near point R$'$, the position of the points E and q is unaltered, it follows that we may differentiate these equations, considering D, d and r, as constant. Accordingly, we have

$$\rho . d\rho - r . d\rho . \sin . \theta - r\rho . \cos . \theta . d\theta = 0,$$
$$\rho' . d\rho' - r . d\rho' . \sin . \theta - r\rho' . \cos . \theta . d\theta = 0.$$

But the cosines of the angles, which the incident and reflected rays make with the tangent to the curve, are, respectively, $\frac{d\rho}{ds}$, and $\frac{-d\rho'}{ds}$, ds being the increment of the arc; and, as these are equal, we have

$$d\rho + d\rho' = 0.$$

If then we substitute $-d\rho$ for $d\rho'$ in the 2d of our equations, we have

$$\rho . d\rho - r . d\rho . \sin . \theta - r\rho . \cos . \theta . d\theta = 0,$$
$$-\rho' . d\rho + r . d\rho . \sin . \theta - r\rho' . \cos . \theta . d\theta = 0;$$

and, multiplying the former by ρ', and the latter by ρ, and subtracting, there is

$$2\rho\rho' - (\rho + \rho') r . \sin . \theta = 0,$$

or,

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2}{r . \sin . \theta}.$$

(85.) When the incident rays are parallel, ρ is infinite, and $\frac{1}{\rho} = 0$; wherefore, denoting the resulting value of ρ' by f, we have

$$f = \frac{r . \sin . \theta}{2}.$$

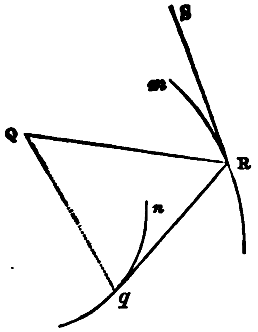
Now, the chord of the osculating circle, passing through the radiant point, $= 2r . \sin . \theta$; wherefore $f =$ one fourth part of this chord. Again, f may be expressed in terms of ρ and θ. For the radius of the osculating circle, expressed as a function of these quantities, is

$$r = \frac{\rho d\rho}{d(\rho . \sin . \theta)}, \quad \therefore f = \tfrac{1}{2} \frac{\sin . \theta . \rho d\rho}{d(\rho . \sin . \theta)}.$$

Finally, if we substitute for $r . \sin . \theta$ its value, $2f$, in the above equation, it becomes

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{f}, \quad \text{whence } \rho' = \frac{\rho f}{\rho - f}$$

(86.) In determining the equation of the caustic curve, we shall use the relation between the radius vector and the angle which it makes with the tangent to the curve; or, which is the same thing, the relation between the distance and perpendicular. It is evident that the reflected ray Rq is a tangent to the caustic nq at the point q: the question therefore is reduced to that of finding the relation between the distance Qq and the angle QqR, that between QR and the angle QRS being given. Now, in the triangle QqR we have the relations



$$\mathrm{Q}q\,.\sin.\,\mathrm{Q}q\mathrm{R} = \mathrm{QR}\,.\sin.\,\mathrm{QR}q,$$

$$\mathrm{Q}q\,.\cos.\,\mathrm{Q}q\mathrm{R} + \mathrm{QR}\,.\cos.\,\mathrm{QR}q = \mathrm{R}q\,;$$

i. e. denoting Qq by u, and the angle QqR by φ,

$$u\,.\sin.\,\varphi = \varrho\,.\sin.\,2\theta,$$
$$u\,.\cos.\,\varphi = \varrho\,.\cos.\,2\theta + \varrho'.$$

Now ϱ' being expressed in terms of ρ and θ, by means of the equations of the preceding article, and one of these quantities then eliminated by means of the equation of the reflecting curve, we shall have two equations containing the other of these variables, together with u and φ; and this variable being eliminated, there results a single equation, containing u and φ only, which is therefore the equation of the caustic.

(87.) As an application of these equations, let us take the case in which the reflecting curve is the *logarithmic spiral*, and the radiant point at its pole.

The property of this curve, with which we are here concerned, is, that the angle made by the radius vector with the tangent is constant. Wherefore, θ being constant in the value of f (85.), we find

$$f = \tfrac{1}{2}\,\frac{\sin.\,\theta\,.\,\varrho d\varrho}{\sin.\,\theta\,.\,d\varrho} = \tfrac{1}{2}\varrho\,;$$

and this being substituted in the equation

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{f},$$

there is $\rho' = \rho$. Now, putting this value of ρ' in equations of the preceding article,

$$u \,.\, \sin.\, \varphi = \rho \,.\, \sin.\, 2\theta,$$
$$u \,.\, \cos.\, \varphi = \rho\,(1 + \cos.\, 2\theta);$$

and, dividing the former by the latter, there is

$$\tan.\, \varphi = \frac{\sin.\, 2\theta}{1 + \cos.\, 2\theta} = \tan.\, \theta.$$

Wherefore, as θ is constant, so is also φ, the angle made by radius vector with the tangent in the caustic curve. That curve is therefore a logarithmic spiral, having the same modulus as the former.

(88.) We now proceed to consider this subject in a more general point of view.

It is evident that when we pass from any reflected ray to that indefinitely near it, the two rays will have, at their intersection (which is the locus of the caustic), the same coordinates; and the only quantities which shall have varied are the coordinates of the reflecting curve. If therefore we differentiate the equation of the reflected ray, relatively to these coordinates only, we shall have two equations involving the coordinates of the intersection, together with those of the reflecting curve and their differentials. If, then, we eliminate the latter from these equations, combined with the equation of the reflecting curve and its differential equations, there will result a single equation, containing only the coordinates of the intersections of the successive reflected rays, and which is therefore the equation of the caustic.

(89.) Thus, let the incident rays be parallel to the axis: making $dy = pdx$, in the equation of the reflected ray (77.), and omitting the traits, it is

$$\rho + (\alpha - x) + p(\beta - y) = 0.$$

Now, if we differentiate this equation relatively to x and y only, and observe that $d\rho = dx$; we have

$$(\beta - y)\, q = p^2,$$

in which $q = \frac{dp}{dx} = \frac{d^2y}{dx^2}$. And if we eliminate x, y, p, and q, from these equations, combined with the equation of the reflecting curve, and its 1st and 2d differentials, we shall obtain a single equation containing α and β only, which is therefore the equation of the caustic.

If we transfer ρ to the other side, in the equation of the reflected ray, then square, and substitute for ρ^2 its value, that equation becomes

$$(p^2 - 1)(\beta - y) + 2p(\alpha - x) = 0,$$

an equation already found (78.). And substituting in this for $(\beta - y)$ its value, $\frac{p^2}{q}$, obtained above, we have

$$(\alpha - x)q = \tfrac{1}{2}p(1 - p^2),$$

which, combined with the equation

$$(\beta - y)q = p^2,$$

will determine the caustic.

(90.) If these equations be squared and added together, there is,

$$[(\alpha - x)^2 + (\beta - y)^2]q^2 = \tfrac{1}{4}p^2(1 + p^2)^2;$$

and, extracting the square root, and dividing by q,

$$\rho = \frac{\frac{1}{2}p(1 + p^2)}{q}.$$

Now, the chord of curvature, in a direction parallel to the axis of abscissæ, is $\frac{2p(1 + p^2)}{q}$; wherefore, denoting this by c, we have

$$\rho = \frac{\mathrm{c}}{4};$$

i. e. the distance of any point in the reflecting curve from the corresponding point in the caustic, or, the focal length of a small oblique pencil of parallel rays is equal to one-fourth of the chord of curvature, taken in a direction parallel to the incident pencil.

(91). As an application of the foregoing theory, let us take the case in which the reflecting curve is a *parabola*, and the rays are incident in a direction *parallel to the axis.*

The equation of the parabola is . . . . $y^2 = ax$,
and differentiating, . . . $2yp = a, \quad p^2 + yq = 0$;
from which there is, . . . $p = \frac{a}{2y}, \quad q = -\frac{a^2}{4y^3}$;
and these values being substituted in equations (89.),
we find . . . $\beta - y = -y$, whence $\beta = 0$,
and . . . $\alpha - x = \frac{a}{4} - x$, whence $\alpha = \frac{a}{4}$.

It appears then, that the caustic is, in this case, reduced to a point, whose coordinates are 0 and $\frac{a}{4}$; that is, the focus of the parabola. Rays parallel to the axis are, therefore, reflected *accurately* to the focus, as is evident from other considerations.

(92). Required the caustic when the reflecting curve is a *parabola* and the rays are incident in a direction *perpendicular to its axis.*

In this case, taking the line perpendicular to the axis, as the axis of abscissæ, the equation of the curve is

$$x^2 = ay.$$

Whence, differentiating twice, we find

$$p = \frac{2x}{a}, \quad q = \frac{2}{a},$$

and substituting in equations (89.),

$$\beta - y = \frac{2x^2}{a}, \quad \therefore \beta = \frac{x^2}{a} + \frac{2x^2}{a} = 3\,\frac{x^2}{a},$$

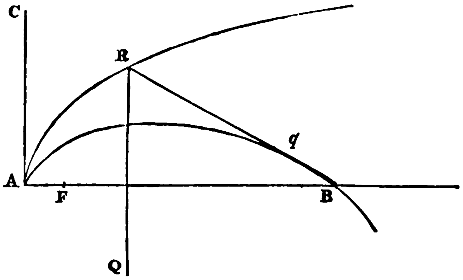
$$\alpha - x = \frac{x}{2}\left(1 - \frac{4x^2}{a^2}\right), \quad \therefore \alpha = x\left(\frac{3}{2} - 2\,\frac{x^2}{a^2}\right).$$

From the former of these equations we get $x^2 = \frac{\beta . a}{3}$, and, substituting in the latter,

$$\alpha = \sqrt{\frac{a}{3}\,\beta}\left(\frac{3}{2} - \frac{2}{3}\,\frac{\beta}{a}\right),$$

the equation of the curve.

The intersections of this curve with the axis are obtained by making $\alpha = 0$, which is satisfied by two values of β, namely, $\beta = 0$, and $\beta = \frac{9}{4}a$. The curve therefore intersects the axis at the vertex A, and at a point, B, whose distance from the vertex AB $= 9$ AF; F being the focus of the parabola. Its form is represented in the annexed figure.



If the distance AB $= \frac{9}{4}a$ be denoted by β', and a thus eliminated from the equation of the curve, that equation assumes the following more simple form:

$$\alpha = \sqrt{\frac{\beta}{3\beta'}}\,(\beta' - \beta).$$

(93.) As another example, we shall take the case in which the reflecting curve is a *cycloid*, and the rays incident in a direction *parallel to its axis*.

If we denote by $2a$ the axis of the cycloid, or the diameter of the generating circle, the relation between the coordinates is expressed by the two equations,

$$x = a\,.\,\text{versin.}\,\theta, \quad y = a(\theta + \sin.\,\theta);$$

and, if we differentiate twice, considering θ as the independent variable, we find

$$dx = a\,.\,d\theta\,.\,\sin.\,\theta, \qquad d^2x = ad\theta^2.\,\cos.\,\theta,$$

$$dy = ad\theta\,.\,(1 + \cos.\,\theta), \qquad d^2y = -\,ad\theta^2.\,\sin.\,\theta.$$

Whence

$$p = \frac{dy}{dx} = \frac{1 + \cos.\,\theta}{\sin.\,\theta}, \quad q = \frac{dx\,.\,d^2y - dy\,.\,d^2x}{dx^3} = -\,\frac{1 + \cos.\,\theta}{a\,.\,\sin.^3\,\theta};$$

and these values being substituted in equations (89.), we obtain the following:

$$\alpha = \tfrac{1}{2}a(3 + \cos.2\theta), \quad \beta = \tfrac{1}{2}a(2\theta - \sin.2\theta);$$

or, making $2\theta = \pi + \theta'$,

$$\alpha = \tfrac{1}{2}a(3 - \cos.\theta'), \quad \beta = \tfrac{1}{2}a(\pi + \theta' + \sin.\theta');$$

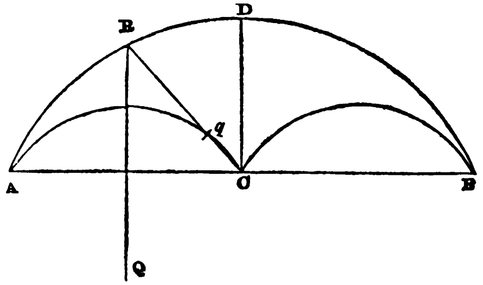
and, finally, if we remove the origin to the point whose coordinates are a and $\tfrac{1}{2}\pi a$, by making

$$\alpha' = \alpha - a, \quad \beta' = \beta - \tfrac{1}{2}\pi . a,$$

we have

$$\alpha' = \tfrac{1}{2}a . \text{versin.}\,\theta', \quad \beta' = \tfrac{1}{2}a(\theta' + \sin.\theta'),$$

the equations of a cycloid, whose axis is a, half the axis of the former, and the coordinates of whose vertex are obtained by bisecting the axis and semibase of the former. The caustic consists, therefore, of two cycloids, each described on half the base of the original cycloid, as is represented in the annexed figure.



(94.) We now proceed to the investigation of the equation of the caustic, when the incident rays diverge from a point.

The equation of the reflected ray, in this case, is

$$\rho'\big[(\alpha - x) + p(\beta - y)\big] + \rho\big[(\alpha' - x) + p(\beta' - y)\big] = 0,$$

in which (α', β') are the coordinates of any point of the reflected ray; and (α, β) those of any point of the incident ray, which are supposed to be given. Then, differentiating this equation relatively to x and y, the coordinates of the reflecting curve, we have

$$\frac{d\rho'}{dx}\big[(\alpha - x) + p(\beta - y)\big] + \rho'\big[(\beta - y)q - (1 + p^2)\big]$$

$$+ \frac{d\rho}{dx}\big[(\alpha' - x) + p(\beta' - y)\big] + \rho\big[(\beta' - y)q - (1 + p^2)\big] = 0,$$

in which $q = \frac{dp}{dx} = \frac{d^2y}{dx^2}$. But since there is

$$(\alpha - x) + p(\beta - y) = -\frac{\rho \, . \, d\rho}{dx},$$

$$(\alpha' - x) + p(\beta' - y) = -\frac{\rho' . \, d\rho'}{dx},$$

this equation becomes

$$\left[\rho'(\beta - y) + \rho\,(\beta' - y)\right] q = (\rho + \rho')\left[1 + p^2 + \frac{d\rho}{dx} \cdot \frac{d\rho'}{dx}\right].$$

Now, since $d\rho + d\rho' = 0$, there is

$$\frac{d\rho}{dx} \cdot \frac{d\rho'}{dx} = -\left(\frac{d\rho}{dx}\right)^2 = -\frac{[(\alpha - x) + p(\beta - y)]^2}{(\alpha - x)^2 + (\beta - y)^2};$$

$$\text{and} \therefore 1 + p^2 + \frac{d\rho}{dx} \cdot \frac{d\rho'}{dx} = \frac{[p(\alpha - x) - (\beta - y)]^2}{(\alpha - x)^2 + (\beta - y)^2}.$$

If, therefore, for the sake of brevity, this function of x be denoted by x^2, the differential equation is written,

$$\left[\rho'(\beta - y) + \rho\,(\beta' - y)\right] q = (\rho + \rho')\,\mathrm{x}^2.$$

Again, if we multiply this equation by p, and eliminate between it and the primitive equation, which is equivalent to

$$\rho'(\alpha - x) + \rho(\alpha' - x) = -p\left[\rho'(\beta - y) + \rho\,(\beta' - y)\right],$$

we obtain

$$\left[\rho'(\alpha - x) + \rho\,(\alpha' - x)\right] q = -p(\rho + \rho')\,\mathrm{x}^2.$$

And, by the aid of these two equations, α' and β', the coordinates of the caustic curve may be completely determined.

(95.) If we transfer the quantities $\rho'(\beta - y)q$, $\rho'(\alpha - x)q$, to the other side of these equations, and then square, and add them together, we find, on reduction,

$$(\rho + \rho')(1 + p^2)\mathrm{x}^2 - 2\rho'\left[(\beta - y) - p(\alpha - x)\right] q = 0.$$

$$\text{or, } (\rho + \rho')(1 + p^2)\mathrm{x} = 2\rho\rho' . q;$$

since $(\beta - y) - p(\alpha - x) = \rho . \mathrm{x}$. And dividing by $\rho\rho'(1 + p^2)\mathrm{x}$,

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{2q}{(1 + p^2)\mathrm{x}}.$$

If the origin of the coordinates be placed at the radiant point, *i. e.* if $\alpha = 0$, $\beta = 0$, then

$$\frac{(1+p^2)\mathrm{x}}{q} = \frac{(px-y)(1+p^2)}{\sqrt{x^2+y^2}\cdot q},$$

which is half the chord of curvature passing through the origin; wherefore, if this chord be denoted by c, there is

$$\frac{1}{\varrho} + \frac{1}{\varrho'} = \frac{1}{\frac{1}{4}c};$$

an equation which has been already obtained by a different method.

From this equation we derive the following proportion:

$$\varrho - \tfrac{1}{4}c : \tfrac{1}{4}c :: \tfrac{1}{4}c : \varrho' - \tfrac{1}{4}c.$$

When the incident rays are parallel, $\frac{1}{\varrho} = 0$, and therefore

$$\varrho' = \frac{c}{4},$$

as has been already shown.

(96.) To determine the equation of the caustic curve, we shall place the origin at the radiant point, or make $\alpha = 0$, $\beta = 0$, in equations (94.), which may be thus written:

$$\varrho q\beta' = (\varrho + \varrho')(qy + \mathrm{x}^2)$$

$$\varrho q\alpha' = (\varrho + \varrho')(qx - p\mathrm{x}^2).$$

But from the equation of the preceding article we obtain

$$\varrho + \varrho' = \frac{2q\varrho^2}{2q\varrho - (1+p^2)\mathrm{x}};$$

and, this value being substituted in the preceding equations, they become:

$$\beta' = \frac{2\varrho(yq + \mathrm{x}^2)}{2\varrho q - (1 + p\mathrm{x}^2)}, \quad \alpha' = \frac{2\varrho(xq - p\mathrm{x}^2)}{2\varrho q - (1+p^2)\mathrm{x}},$$

in which $\varrho = \sqrt{x^2+y^2}$, $\mathrm{x} = \dfrac{px-y}{\sqrt{x^2+y^2}}$.

To obtain the equation of the caustic, x, y, p, and q, are to be eliminated from these two equations, combined with the equation of the reflecting curve, and its derivatives; the resulting equation, containing α' and β' only, will be that of the caustic.

(97.) The relation between the coordinates of the caustic and those of the reflecting curve, however, may be exhibited under a somewhat simpler form: for if, in the differential equation of the reflected ray

$$\left[\rho(\beta' - y) - \rho' y\right] = (\rho + \rho')\mathrm{x}^2,$$

we substitute for ρ' its value,

$$\rho \, . \, \frac{(\alpha' - x) + p(\beta' - y)}{x + py},$$

we find

$$(\beta x - \alpha y) q = (\alpha + p\beta)\mathrm{x}^2;$$

in which we have omitted the traits over α and β, as no longer necessary. This equation, combined with that of the reflected ray (78.), namely,

$$(\beta - y)\left[(p^2 - 1)x - 2py\right] + (\alpha - x)\left[(p^2 - 1)y + 2px\right] = 0,$$

will serve conveniently for the determination of the caustic curve.

(98.) Before we proceed to apply this theory to any particular examples, we shall investigate some other general properties of the caustic curve.

As this curve is the locus of the intersections of the successive reflected rays, it is evident that each reflected ray is a tangent to the curve. Now the tangent of the angle, which the tangent to the curve makes with the axis of abscissæ, is $\frac{d\beta'}{d\alpha'}$; therefore, ω' being the angle which the reflected ray makes with the same axis, $\frac{d\beta'}{d\alpha'} = \tan. \omega'$, or,

$$d\beta' . \cos. \omega' - d\alpha' . \sin. \omega' = 0.$$

(99.) To obtain another equation between $d\alpha'$ and $d\beta'$, we shall differentiate the equation

$$(\beta' - y)^2 + (\alpha' - x)^2 = \rho'^2,$$

by which means we have

$$(\beta' - y)(d\beta' - dy) + (\alpha' - x)(d\alpha' - dx) = \rho' . d\rho';$$

or, putting for $(\beta'-y)$, $(\alpha'-x)$, their values, $\varrho'.\sin.\omega'$, $\varrho'.\cos.\omega'$,

$$(d\beta' - dy)\sin.\omega' + (d\alpha' - dx)\cos.\omega' = d\varrho'.$$

Again, performing the same operation on the equation

$$(\beta - y)^2 + (\alpha - x)^2 = \varrho^2,$$

we find

$$-dy.\sin.\omega - dx.\cos.\omega = d\varrho;$$

and, finally, adding these equations, and remarking that

$$dy.\sin.\omega + dx.\cos.\omega + dy.\sin.\omega' + dx.\cos.\omega' = 0,$$

there is

$$d\beta'.\sin.\omega' + d\alpha'.\cos.\omega' = d\varrho + d\varrho'.$$

Now, if we square this equation, and that of the preceding article, and add them together, we find

$$d\beta'^2 + d\alpha'^2 = (d\varrho + d\varrho')^2 = dz^2,$$

dz being the differential of the arc of the caustic curve;

$$\therefore\ dz = d\varrho + d\varrho',$$

$$\text{and } z = \varrho + \varrho' + \text{const.}$$

When the reflecting curve, therefore, is an algebraic curve, its caustic is always *rectifiable.*

(100.) As an application of the preceding theory, we shall investigate the equation of the caustic curve when rays diverging from any point are reflected by a spherical surface.

If δ be the distance of the radiant point from the centre of the circle, its equation is

$$y^2 + (x - \delta)^2 = r^2;$$

deriving from this equation the values of p and q, and substituting them in equations (96.), we should have α and β, the coordinates of the caustic curve, expressed in terms of x and y, the coordinates of the reflecting curve; and eliminating the latter by means of these equations and that of the circle above written, we should have a single equation, containing α and β only. Having already, however, obtained the equation of the reflected ray in this case, we shall save trouble by employing it in the present investigation. That equation, we have found, is

$$\beta[\delta.\cos.2\omega + r.\cos.\omega] - \alpha[\delta.\sin.2\omega + r.\sin.\omega] + \delta r.\sin.\omega = 0;$$

and differentiating with respect to ω only, there is

$$\beta[2\delta.\sin.2\omega + r\sin.\omega] + \alpha[2\delta.\cos.2\omega + r.\cos.\omega] - \delta r.\cos.\omega = 0;$$

and if we eliminate α and β successively from these equations, we shall find

$$\beta = \frac{2r\delta^2.\sin.^3\omega}{r^2 + 3\delta r.\cos.\omega + 2\delta^2}, \quad \alpha = \frac{\delta r[r + \delta.\cos.\omega(1 + 2\sin.^2\omega)]}{r^2 + 3\delta r.\cos.\omega + 2\delta^2};$$

and, finally, eliminating ω between these equations, we shall have an equation containing α and β only, which will be consequently the equation of the caustic curve.

This elimination cannot be effected except in some particular cases, in which the value of δ simplifies the above formulæ. If, for example, the radiant point be at the centre of the circle, or $\delta = 0$, we find $\beta = 0$, $\alpha = 0$; and the caustic is reduced to a point, namely the centre itself, as is otherwise evident.

(101.) When the radiant point is infinitely distant, or the incident rays parallel, δ is infinite, and the above values become

$$\beta = r.\sin.^3\omega, \quad \alpha = r.\cos.\omega\left(\tfrac{1}{2} + \sin.^2\omega\right).$$

The elimination is easily effected in this case; for, squaring these equations and adding, we find

$$\beta^2 + \alpha^2 = \frac{r^2}{4}(1 + 3.\sin.^2\omega),$$

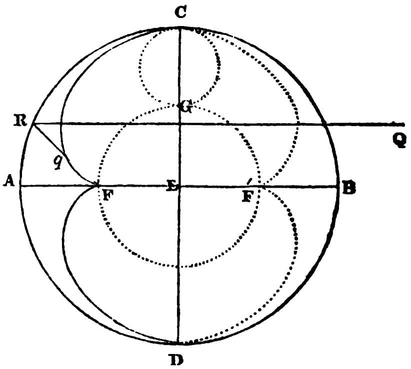
$$\text{or,} \quad r^2.\sin.^2\omega = \tfrac{4}{3}\left(\beta^2 + \alpha^2 - \frac{r^2}{4}\right);$$

and substituting for sin. ω its value derived from the former of the two equations,

$$r^{\frac{4}{3}}.\beta^{\frac{2}{3}} = \tfrac{4}{3}\left(\beta^2 + \alpha^2 - \frac{r^2}{4}\right),$$

the equation of the caustic in this case.

This equation is that of an *epicycloid*, generated by the revolution of a circle whose radius is $\frac{r}{4}$, on another whose radius is $\frac{r}{2}$; the latter being concentric with the reflecting circle. The form of the caustic is that represented in the adjoining figure; in which ABCD is the reflecting circle, FF′ and CG the *base* and *generating circle* of the epicycloid; the radius of the former being one half, that of the latter one fourth of the radius of the reflecting curve. The branch CFD of the caustic is generated by reflexion at the concave surface CAD; the branch CF′D, by reflexion at the convex surface CBD; the latter, however, being *imaginary*, in the sense in which we have used that word as applied to foci. The curve has a cusp at F and F′, the middle points of the two radii, which are the principal foci of the rays incident on the two hemispheres.



This curve may be familiarly exhibited by exposing a glass, full of milk, to the rays of the sun: the caustic will be defined by a bright line of light on the surface of the liquid.

(102.) When the radiant point is at the extremity of the diameter, $\delta = r$; and the values of β and α become

$$\beta = \tfrac{2}{3} r . \sin. \omega \, (1 - \cos. \omega),$$

$$\alpha = \tfrac{1}{3} r + \tfrac{2}{3} r . \cos. \omega \, (1 - \cos. \omega).$$

The resulting equation will be more simple if we employ polar coordinates, first transferring the origin to the point whose abscissa is $\frac{1}{3}r$. Thus, let

$$\beta = \varrho . \sin. \theta, \qquad \alpha = \tfrac{1}{3} r + \varrho . \cos. \theta,$$

and we have

$$\varrho . \sin. \theta = \tfrac{2}{3} r . \sin. \omega \, (1 - \cos. \omega),$$

$$\varrho . \cos. \theta = \tfrac{2}{3} r . \cos. \omega \, (1 - \cos. \omega);$$

and if we square these equations, and add them, we find

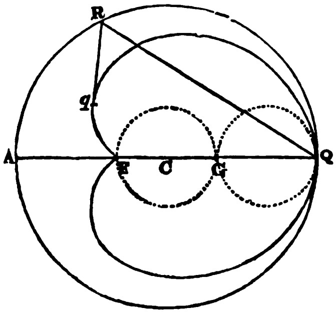
$$\varrho = \tfrac{2}{3} r \, (1 - \cos. \omega);$$

and dividing the latter of the two equations by this, we get $\cos. \omega = \cos. \theta$; whence, finally,

$$\rho = \tfrac{2}{3} r (1 - \cos. \theta),$$

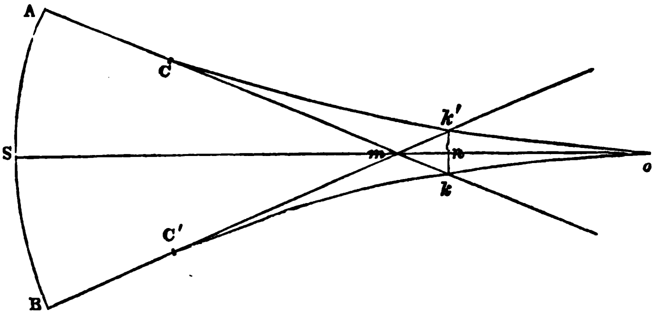
the equation of the *cardioide*, or the epicycloid generated by the revolution of a circle upon another of equal radius.

The form of the curve is represented in the adjoining figure, in which QRA is the reflecting circle, Q*q*F the epicycloid generated by the circle QG rolling upon the circle FG, which is concentric with the reflecting circle; the radius of the generating circle and base being each one third of that of the reflecting circle.



(103.) The caustic being the locus of the intersections of the successive reflected rays, it is evident that the condensation of the light in this curve is infinitely greater than in the surrounding space. Again, this curve has always a cusp corresponding to the intersection of the central rays with the axis: at this point the number of rays, collected into a given space, is infinitely greater than at any other point of the curve, and the point itself is the geometric focus of rays indefinitely near the axis in their incidence. Now, the interval between this point and the intersection of any other reflected ray with the axis is the longitudinal *aberration* of that ray; and the smallest space through which all the reflected rays pass is the *least circle of aberration* or diffusion.

The consideration of the magnitude of this circle is of considerable importance, inasmuch as it determines the limit of the *concentrating power* of any reflecting surface. To determine it, let C*o*, C'*o*, be the branches of the caustic; AC*m*, BC'*m*, the extreme rays, touching the caustic at the points C and C', and cutting it again at *k* and *k'*. Then, since all the reflected rays are tangents to the caustic, in some part of its length C*o* or C'*o*; it is evident that they must all pass through *kk'*, or that *kk'* is the diameter of the least circle of aberration, and *nk* its



radius. Consequently, the radius of the least circle of aberration is the ordinate of the intersection of the extreme ray with the caustic curve, and is accordingly obtained by eliminating α between the equation of that extreme ray and that of the caustic; the resulting value of β is the quantity sought.

It is to be observed, that the final equation which gives the ordinate of the point *k*, or the radius of the least circle of aberration, must also give the ordinate of the point c, which is also a point of intersection of the extreme ray with the caustic. Moreover, as the extreme ray is a tangent to the curve at this point, the point c is a *double point*. Accordingly, the resulting equation must have two equal roots, belonging to the ordinates of the point c, and independent of the required values of β; and these roots being known and the equation depressed, the resulting equation will give the required values of β.

(104.) To apply this to an example: let us seek the least circle of aberration, when parallel rays are reflected by a spherical surface.

Making δ infinite in equation (81.), the equation of the reflected ray in this case is

$$\beta \, . \cos . \, 2\omega - \alpha \, . \sin . \, 2\omega + r \, . \sin . \, \omega = 0.$$

Whence, substituting for $\sin . \, 2\omega$ and $\cos . \, 2\omega$ their values expressed in terms of $\sin . \, \omega$, and denoting the extreme value of $\sin . \, \omega$ by a, we have for the extreme ray

$$\alpha = \frac{\beta(1 - 2a^2) + ra}{2a\sqrt{1 - a^2}} \, ;$$

and this value being substituted in equation (101.), the equation of the caustic curve in this case, there is

$$3a^2(1 - a^2)(r^2\beta)^{\frac{2}{3}} = \beta^2 + 2\beta r . a(1 - 2a^2) + r^2a^4.$$

Finally, if we make $\beta = rx^3$, x being another assumed variable, r will disappear from the resulting equation, which thus becomes

$$x^6 + 2x^3 . a(1 - 2a^2) - 3x^2 . a^2(1 - a^2) + a^4 = 0.$$

Now, this equation must have two equal roots belonging to the ordinates of the point C, at which the extreme ray touches the caustic. But since the general value of β is $r \sin.^3 \omega$ (101), for this point there is

$$\beta = ra^3, \text{ and therefore } x = a.$$

Accordingly, the preceding equation must have two roots equal to a, and is therefore divisible by $(x - a)^2$. Performing the division, the depressed equation is

$$x^4 + 2ax^3 + 3a^2x^2 + 2ax + a^2 = 0;$$

an equation which gives the values of x, and therefore of $\beta = rx^3$, the radius of the least circle of aberration.

This is the general solution of the problem, whatever be the aperture of the mirror. When a is very small, x is so likewise: in approximating, therefore, to its value, we may neglect all the terms of this equation but the two last, the rest being, in this case, indefinitely small in comparison. We have, therefore,

$$2x + a = 0, \quad \text{or, } x = -\tfrac{1}{2}a;$$

$$\therefore \ \beta = rx^3 = -\tfrac{1}{8}ra^3;$$

a result which agrees with that already obtained (66.).

CHAPTER V.

OF LIGHT REFRACTED AT PLANE SURFACES.

I.

Of Refraction at a single plane Surface.

(105.) WHEN a ray of light passes from one medium into another of a different nature, it is, in general, bent from its original direction. This modification of light is called *refraction.*

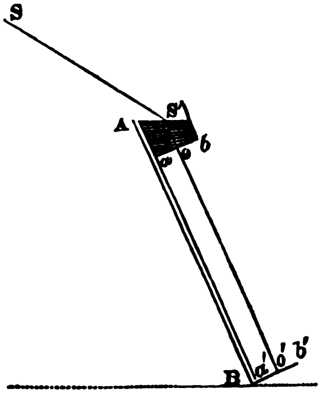
The two portions of the ray, before and after incidence upon the surface bounding the two media, are called the *incident* and *refracted rays.* The angles which they make with the perpendicular to the refracting surface at the point of incidence, are called the *angles of incidence* and *refraction,* respectively; and the angle, contained by the refracted ray with the incident ray produced, is called the *angle of deviation.*

In general, when a ray passes from a rarer into a denser medium, the angle of incidence is greater than the angle of refraction; and, therefore, the deviation takes place *towards* the perpendicular at the common surface. When the passage of the ray, on the contrary, is from the denser into the rarer medium, the reverse takes place, and the deviation is *from* the perpendicular.

The angle of deviation, corresponding to a given angle of incidence, is, in general, different for different media; and any medium is said to have a greater or less *refracting power* than another, according as the deviation produced by it, at a given incidence, is greater or less than that of the other.

(106.) *The angles which the portions of the ray in the two media make with the perpendicular to the refracting surface at the point of incidence, are in the same plane, and their sines are in an invariable ratio, whatever be the direction of the incident light.*

This law, we have already observed, is the result of experience; and we have explained an experiment by which its truth may be familiarly exhibited. However, as it is the foundation of the whole theory of refracted light, it will not be amiss to show in what manner the constant ratio of the sines may be more strictly established.



Let a perfectly straight beam, AB, be provided; to which are attached, near the extremities, two plates, *ab*, *a'b'*, so that their planes are perpendicular to the line of the beam, and therefore parallel. Now, the upper of these plates being perforated in any point, *o*, let a point, *o'*, be marked on the lower, exactly corresponding in position. Then, a glass vessel, whose bottom is perfectly plane, being placed on the upper plate, and partly filled with water, or any other transparent fluid, let a beam of the sun's light, SS', be incident on the surface of the fluid at S'; and let the position of the beam be shifted, until the refracted ray, S'*o*, passing through the aperture at *o*, shall meet the other plate in the corresponding point *o'*; and, at that moment, let the altitude of the sun be taken, and the inclination of the beam to the horizon measured.

Now, the surface of the fluid being horizontal, the altitude of the sun is equal to the angle which the incident ray makes with the surface of the fluid, or to the complement of the angle of incidence. Also, the refracted ray *oo'* being evidently parallel to the beam, the inclination of the beam to the horizon is equal to the angle which the refracted ray makes with the surface of the fluid, or, to the complement of the angle of refraction.

Thus, the experiment being repeated at different altitudes of the sun, we obtain several angles of incidence with their corresponding angles of refraction; and their sines, being taken from the trigonometrical tables, will be found always in the same ratio.

In the foregoing experiment, the course of the ray is from air into water. It would be easy to modify the experiment so as to adapt it to the case in which the light proceeds in the opposite direction. As the fullest confirmation of these laws, however, is derived from the agreement of the results deduced from them with experience, it is unnecessary to dwell longer upon the subject.

(107.) If θ and θ' be used to denote the angles which the portions of the ray in the rarer and denser medium, respectively, make with the perpendicular to the common surface at the point of incidence, the law of refraction will be expressed by the equation

$$\frac{\sin.\,\theta}{\sin.\,\theta'} = \mu,$$

μ being a constant quantity.

This constant is termed the *index of refraction*: and, since θ is greater than θ' (105.), it is evident that μ is always greater than unity.

The index of refraction, though constant for the same media, is yet, as we have already remarked, different for different media. The least value of μ is unity, its value when a ray proceeds out of any medium into another of equal density: in this case, it is evident that the ray will undergo no refraction. The greatest known value of μ is 3; which is its value when a ray proceeds from a vacuum into chromate of lead; and between these extreme values it is found of every intermediate magnitude. Thus: for water, $\mu = 1{,}336$. For crown glass, $\mu = 1{,}535$. For flint glass, $\mu = 1{,}6$. For diamond, $\mu = 2{,}487$. For chromate of lead, $\mu = 3$.

In all these cases, in which a single medium only is noticed, the other is understood to be a *vacuum*. The index of refraction, in such cases, is called the *absolute index* of refraction of the medium.

(108.) The angle of deviation is the difference between the angles of incidence and refraction.

It may easily be shown that, the greater the incidence, the greater must also be the deviation. For, by the equation of the preceding article,

$$\sin.\theta = \mu \sin.\theta'.$$

Now, if we add and subtract $\sin.\theta'$ on both sides of this equation, and divide the latter result by the former, there is

$$\frac{\sin.\theta - \sin.\theta'}{\sin.\theta + \sin.\theta'} = \frac{\mu - 1}{\mu + 1};$$

or, since the first member of this equation $= \dfrac{\tan.\frac{1}{2}(\theta - \theta')}{\tan.\frac{1}{2}(\theta + \theta')}$,

$$\tan.\tfrac{1}{2}(\theta - \theta') = \frac{\mu - 1}{\mu + 1}.\tan.\tfrac{1}{2}(\theta + \theta').$$

From which it appears that, as θ, and therefore also θ', increases; $\theta - \theta'$, the deviation, likewise increases.

When the angles of incidence and refraction are very small, they may, without sensible error, be considered proportional to their sines; and we have

$$\theta = \mu\theta'.$$

$$\therefore\ \theta - \theta' = (\mu - 1)\,\theta' = \frac{\mu - 1}{\mu}\,\theta.$$

That is, the deviation of a ray, which passes nearly perpendicularly through a refracting surface, bears a given ratio to the angle of incidence.

(109.) It follows from the equation $\sin.\theta = \mu.\sin.\theta'$, that θ', the angle which the portion of the ray in the denser medium makes with the perpendicular to the surface, can never exceed a certain limit. For, it is evident that $\sin.\theta'$, and therefore θ' itself, is greatest when $\sin.\theta$ is so. But the greatest value of $\sin.\theta$ is unity; to which it is equal, when θ is a right angle. Wherefore, if $\theta_{\prime}$ denote the corresponding value of θ',

$$\sin.\theta_{\prime} = \frac{1}{\mu}.$$

Thus, when the refraction takes place at the surface of water,

$$\mu = 1{,}336, \quad \text{and} \therefore \ \theta_{\prime} = 48^\circ.28'.$$

When the refracting medium is crown glass,

$$\mu = 1{,}535, \quad \text{and} \therefore \ \theta_{\prime} = 40^\circ.39'.$$

Now θ' is the angle of refraction or incidence, according as the course of the ray is from the rarer into the denser medium, or in the opposite direction. In the latter case, therefore, *i. e.* when the course of the ray is from the denser into the rarer medium, it cannot be transmitted when the angle of incidence exceeds a certain limit, $\theta_{\prime}$, whose sine $= \frac{1}{\mu}$. In fact, it appears from experiment, that the ray whose incidence exceeds this limit is turned back into the denser medium, being *reflected*, and not refracted, at the common surface.

This seems to afford a simple method of determining the index of refraction of any substance: for we have only to observe the limiting angle, at which a ray of light ceases to emerge from that substance into air; the reciprocal of the sine of this angle is the index of refraction between air and that substance.

(110.) In what has preceded, we have used the character μ to denote the ratio $\frac{\sin.\theta}{\sin.\theta'}$, in which θ and θ' are the angles which the branches of the ray in the rarer and denser medium, respectively, make with the perpendicular to the surface at the point of incidence. In this case μ is the same, whether the course of the ray is from the rarer into the denser medium, or *v. v.*; and is always greater than unity. As it is frequently necessary, however, that our analysis should designate the direction of the light, we shall employ the symbol m to denote the ratio of the sines of the angles of incidence and refraction; so that

$$m = \frac{\sin.(\text{inc.})}{\sin.(\text{ref.})}.$$

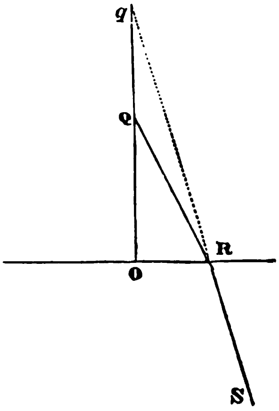
Accordingly, when the course of the ray is from the rarer into the denser medium, $m = \mu$; when in the opposite direction, $m = \frac{1}{\mu}$.

(111.) If a pencil of parallel rays be incident on a plane refracting surface, the refracted rays are also parallel.

For, since the incident rays are parallel, as also the perpendiculars to the surface at the points of incidence, it follows that the planes of incidence are parallel, and the angles of incidence equal. Hence, since the angles of incidence and refraction are in the same plane, and their sines in an invariable ratio, the planes of refraction are parallel, and the angles of refraction equal. Accordingly, the refracted rays must be parallel, as they lie in parallel planes, and contain equal angles with parallel lines in those planes, namely, the perpendiculars to the surface at the points of incidence.

(112.) A pencil of rays, diverging from a point, being incident nearly perpendicularly upon a plane refracting surface, it is required to determine the focus of the refracted pencil.

Let Q be the radiant point, QO the perpendicular from it on the refracting surface; QR any incident ray, which is refracted in the direction RS; and let this refracted ray be produced backwards to meet the perpendicular to the refracting surface in q. Then, in the triangle QRq, there is



$$\frac{\mathrm{R}q}{\mathrm{RQ}} = \frac{\sin.\ \mathrm{RQ}q}{\sin.\ \mathrm{R}q\mathrm{Q}} = \frac{\sin.\ \mathrm{RQO}}{\sin.\ \mathrm{R}q\mathrm{O}}.$$

But RQO and RqO are equal to the angles of incidence and refraction respectively, and the ratio of their sines $= m$; wherefore

$$\frac{\mathrm{R}q}{\mathrm{RQ}} = m.$$

Now, since the incident rays are, by supposition, *quam proximè* perpendicular to the refracting surface, the point R is indefinitely near to O; and, ultimately, when R coincides with O, the distances RQ and Rq coincide with OQ and Oq; wherefore, if these latter distances be denoted by δ and δ', there is

$$\frac{\delta'}{\delta} = m;$$

that is, the distance of the focus of the refracted rays from the surface, is to the distance of the focus of incident rays from the same, in the constant ratio of the sine of incidence to the sine of refraction.

When the radiant point is in the rarer medium, $m = \mu$, and is greater than unity, wherefore $\delta' > \delta$. On the contrary, when the radiant point is in the denser medium, $m = \frac{1}{\mu}$, and is less than unity, and therefore $\delta' < \delta$.

The distance δ', thus determined, is the *ultimate* value of the distance of the intersection of the refracted ray with the axis from the surface, and to which it approaches more and more nearly as the aperture OR is diminished. But, as it is evident that the number of rays diverging from the point Q, and incident on a given breadth of the surface, is greatest at the point O; so the number of refracted rays collected into a given space on the perpendicular OQ, is greatest at the corresponding point of that perpendicular, whose distance is determined above. This point, therefore, is justly considered as the focus of the refracted pencil. This subject will be placed in a clearer and more general point of view when we treat of caustics.

(113.) Let us now seek the intersection of any refracted ray whatever with the axis.

Let the distances OQ and Oq be denoted by δ and δ', and the semiaperture OR by α; then, if we square the equation Rq $= m \,.$ RQ, and substitute for Rq and RQ their values expressed in terms of δ, δ' and α, there is

$$\delta'^2 + \alpha^2 = m^2 (\delta^2 + \alpha^2):$$

$$\therefore\ \delta' = [m^2\delta^2 + (m^2 - 1)\alpha^2]^{\frac{1}{2}}.$$

When the aperture is small, we may obtain from this an approximate value, which will be sufficiently near the truth for all practical purposes. For, if the preceding value of δ' be developed, and all the powers of α, beyond the second, neglected, there is

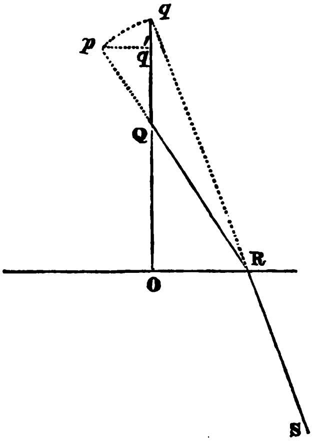
$$\delta' = m\delta + \tfrac{1}{2}(m^2 - 1)\frac{\alpha^2}{m\delta}.$$

But when $\alpha = 0$, there is $\delta' = m\delta$, the approximate value already found; wherefore, if we denote this latter by $\delta_{\prime}$, and subtract,

$$\delta' - \delta_{\prime} = \tfrac{1}{2}(m^2 - 1)\frac{\alpha^2}{m\delta}.$$

This quantity is called the aberration of the extreme ray.

The intersection of any refracted ray with the axis may be exhibited by a simple construction.



QR being any incident ray whatever, proceeding from the point Q, let it be produced backwards, until it meet the perpendicular to the axis, $q'p$, erected at q', the focus of rays indefinitely near the axis; then, if with the centre R and radius Rp a circle be described, cutting the axis in q, that point will be the intersection of the refracted ray with the axis. For, joining Rq, there is

$$\frac{\mathrm{R}q}{\mathrm{RQ}} = \frac{\mathrm{R}p}{\mathrm{RQ}} = \frac{\mathrm{O}q'}{\mathrm{OQ}} = m;$$

whence Rq is the refracted ray, and qq' the aberration.

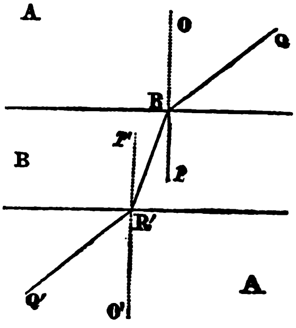
(114.) It is necessary to observe that, when a beam of light falls upon the surface of a refracting medium, the whole of the incident light is, in no case, transmitted. A portion is always reflected; and, when the course of the light is from the rarer into the denser medium, the proportion which the reflected bears to the transmitted part increases regularly with the incidence; being smallest when the light is incident perpendicularly, and greatest when the light just grazes the surface, neither portion ever vanishing.

The case is somewhat different when the light proceeds from the denser into the rarer medium: in this case the reflected portion increases regularly with the incidence, as before, until the angle of incidence reaches the limiting value above mentioned; when, suddenly, the *whole* of the incident light is reflected, and none whatever transmitted. This limiting angle has been, accordingly, called the *angle of total reflexion.* This reflexion, being total, far exceeds that which takes place at the most highly polished reflecting surfaces, the most brilliant of which absorb a considerable portion of the incident light.

II.

Of Refraction at parallel plane Surfaces.

(115.) When a ray of light traverses a medium bounded by parallel planes, and re-enters the original medium, the emergent ray is parallel to the incident.



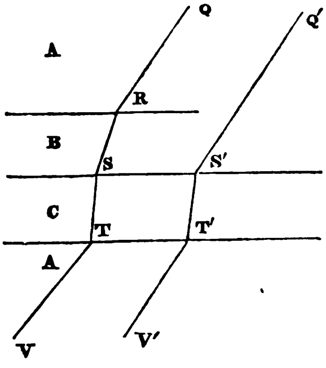
Let QRR'Q' be the course of the ray, passing from the medium A into the medium B at R, and re-emerging into the former at R'. Then, by the law of refraction, as it has been laid down (106.), the ratio of the sines of the angles, which the branches of the ray in the media A and B form with the perpendicular to the bounding surface, is constant, whatever be the course of the ray; that is, if OP and O'P' be the perpendiculars to the parallel surfaces at the points of incidence and emergence,

$$\frac{\sin.\ QRO}{\sin.\ R'Rp} = \frac{\sin.\ Q'R'O'}{\sin.\ RR'p'}.$$

But, on account of the parallelism of the surfaces, the angles $R'Rp$ and $RR'p'$ are equal: therefore the angles QRO, Q'R'O', are also equal, and the lines QR, Q'R', therefore, parallel.

(116.) When a ray of light, passing through the second medium, enters a third which is different from either of the former, the surfaces bounding the media being parallel, the course of the ray in the third medium will be the same as if the light had entered it directly from the first, and is therefore altogether independent of the intermediate medium.

For the demonstration of this law, which may be considered as the foundation of all that relates to successive refraction by different media, we must resort to experience. It is proved by experiment that if a ray of light, QRSTV, passing from the medium A, traverse two others, B and C, and finally emerge into the original medium, all the bounding surfaces being parallel, the emergent ray TV will be parallel to the incident QR.



Now, if we suppose another ray, Q′S′T′V′, to be incident directly from the first upon the third medium, in a direction parallel to QR, and passing through the third to re-enter the first, it follows from the last article that the incident and emergent rays, Q′S′ and T′V′, are also parallel. Hence, the incident rays, QR and Q′S′, being supposed parallel, the emergent rays, TV and T′V′, are also parallel; and, as they are both equally refracted at the common surface of the same media C and A, ST and S′T′, the branches in the medium C, are likewise parallel.

Hence, when a ray of light passes from any medium, through an intermediate medium, into a third, all being bounded by parallel surfaces, the total deviation of the ray is the same as if it had passed directly from the first into the third.

This principle may be generalized: for, since the course of the ray in the third medium is the same as if the light had passed directly into it from the first, it follows, that if a fourth medium be added to the other three, the course of the ray in it will be the same as if the light had passed from the first into the third, and thence into the fourth; which is the same as if the light had passed *directly* from the first into the fourth. And, generally, whatever be the number of media, the course of the ray in the last, and therefore the total deviation, will be the same as if the light had been incident directly from the first into the last, parallel to its original direction.

(117.) The analytic expression of these laws is as follows:

Let I' and R' denote the angles of incidence and refraction from the first medium into the second; I'' and R'', from the second into the third; and $\text{I}_{,}$ and $\text{R}_{,}$ from the third into the first. Then, if m', m'', and $m_{,}$, denote the corresponding ratios of the sines,

$$m' = \frac{\sin.\text{I}'}{\sin.\text{R}'}, \quad m'' = \frac{\sin.\text{I}''}{\sin.\text{R}''}, \quad m_{,} = \frac{\sin.\text{I}_{,}}{\sin.\text{R}_{,}}.$$

And if we multiply these equations together, and observe that, on account of the parallelism of the surfaces, $\text{I}'' = \text{R}'$, $\text{I}_{,} = \text{R}''$, there is

$$m'm''m_{,} = \frac{\sin.\text{I}'}{\sin.\text{R}_{,}}.$$

But by experiment it appears that the emergent ray is parallel to the incident, and therefore $\text{R}_{,} = \text{I}'$; whence we have

$$m'm''m_{,} = 1.$$

Now, if m denote the ratio of the sines out of the first medium into the third, $m = \frac{1}{m_{,}}$; wherefore there is

$$m = m'm''.$$

That is, the ratio of the sines from the first medium into the third, is the product of the corresponding ratios from the first into the second, and from the second into the third.

This conclusion is readily generalized: for, if m''' denote the ratio of the sines from the third medium into a fourth, then the ratio from the first into the fourth $= mm''' = m'm''m'''$. And, in general, whatever be the number of successive media, if m denote the index of refraction from the first into the last, we have

$$m = m'm''m'''m'''', \&c.$$

This result is of great importance; for by means of it we can obtain the measure of refraction between any two media, provided we are able to connect them by any intermediate media whatever, at whose common surfaces the refraction is known.

(118.) In the case of three media, if the second or intermediate medium be a vacuum, $m' = \frac{1}{\mu'}$, and $m'' = \mu''$; μ' and μ'' denoting the *absolute* indices of refraction of the extreme media: accordingly there is

$$m = \frac{\mu''}{\mu'}.$$

That is, the *relative* index of refraction between any two media is equal to the quotient arising from the division of the absolute index of the second by that of the first.

Thus, if it be required to find the relative index of refraction from water into glass, there is

$$\mu' = \tfrac{4}{3}, \text{ and } \mu'' = \tfrac{3}{2}, \text{ nearly,}$$

in which μ' and μ'' denote the absolute indices of refraction of water and glass, respectively. Wherefore

$$m = \frac{\mu''}{\mu'} = \tfrac{9}{8}, \text{ nearly.}$$

(119.) When a pencil of rays, diverging from a point, is incident nearly perpendicularly upon a medium bounded by parallel plane surfaces, and emerges into the original medium, it is required to find the focus of the emergent pencil.

Let δ denote the distance of the focus of the incident pencil from the first surface; δ' and δ'', the distances of the conjugate foci from the same after the first and second refraction, respectively: then, if θ denote the thickness of the medium, $\delta' + \theta$ and $\delta'' + \theta$ will be the distances of the conjugate foci, at the second refraction, from the refracting surface; and we have the equations:

$$\delta' = m\delta,$$

$$\delta' + \theta = m(\delta'' + \theta);$$

and subtracting the former from the latter, there is

$$\theta = m(\delta'' - \delta) + m\theta.$$

$$\therefore \quad \delta - \delta'' = \frac{m-1}{m}\theta.$$

From which it appears that the focus of the pencil is brought nearer to the medium by the double refraction, the distance between the foci of the incident and emergent pencils being to the thickness of the medium in a constant ratio.

If the rays meet with a second medium bounded by planes parallel to those of the former, the focus of the pencil emergent from the first medium will, after refraction by the second, be brought nearer to it by the quantity

$$\frac{m'-1}{m'}\theta',$$

in which θ' denotes the thickness of the second medium, and m' the index of refraction from the surrounding medium into it; wherefore the interval between the first and last focus will be

$$\frac{m-1}{m}\theta + \frac{m'-1}{m'}\theta':$$

and this is so, whether the media are in contact, or separated by any interval whatever.

In the same manner, if there be several such media, their combined effect on the place of the focus of the emergent pencil is obtained by adding together the separate effects of each.

III.

Of Refraction by Prisms.

(120.) We now proceed to consider the refraction of a ray, or system of rays, in passing through a medium bounded by plane surfaces which are inclined to one another at any angle. We shall confine our attention to the most important case: that, namely, in which the refraction takes place in a plane perpendicular to both the bounding surfaces, and therefore to the right line in which they meet.

Any medium bounded by two plane surfaces, which are inclined to one another at any angle, is called in optics a *prism.*

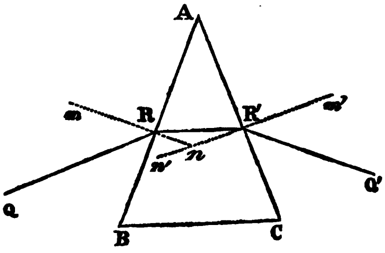
The two surfaces are called the *faces* of the prism; the line in which they meet, the *edge;* and the angle which they form, the *refracting angle.*

Any section of the prism, formed by a plane perpendicular to both surfaces, and therefore also to the edge of the prism, is called a *principal section* of the prism.

(121.) When a ray of light passes through a prism, which is denser than the surrounding medium, the total deviation of the ray is, in all cases, *from the refracting angle.*

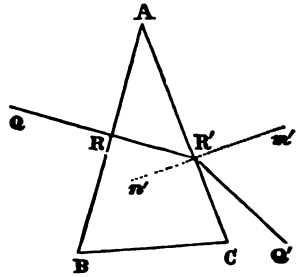
Let BAC, in the adjoining figures, represent a principal section of the prism; $QRR'Q'$ the course of the refracted ray, which is obviously contained in the plane of that section; and let mn, $m'n'$, be the perpendiculars to the faces of the prism at the points of incidence and emergence. There are, then, three cases to be considered, inasmuch as the angles ARR', $AR'R$, which the ray within the prism makes with the sides towards the vertex, may be both acute, one right, or one obtuse.

In the first case, it is evident that the incident and emergent rays, QR, $R'Q'$, lie on the sides of the perpendiculars *from the vertex;* and, since the branch of the ray in the rarer medium con-

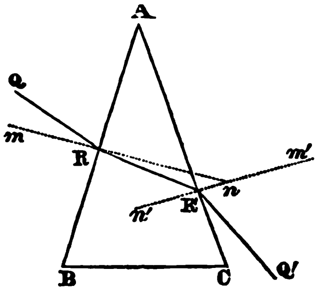


tains a greater angle with the perpendicular than that in the denser, the deviation, both at ingress and egress, will be from the vertex of the prism.

In the second case, namely, when one of the angles, ARR', AR'R, is right, and the other acute, there will be no deviation whatever at the former; and at the latter, the deviation will be, for the reason already given, from the vertex of the prism.



In the third case, where one of the angles, ARR', is obtuse, and the other, AR'R, acute, since the ray RQ will evidently lie at the side of the perpendicular towards the vertex, and R'Q' at the opposite side, the deviation at R will be towards the vertex, and at R' from it. But the angle of incidence at R' is greater than at R; the former being the exterior, the latter the interior angle of the triangle RR'n: hence the deviation in the former case is greater than in the latter (108.), and therefore the *total* deviation, which in this case is the difference of the partial deviations, will be from the vertex.



(122.) The vertical angle of the prism is equal to the sum or difference of the angles which the course of the ray within the prism makes with the perpendiculars to the faces at the points of incidence and emergence.

For, if ψ and ψ' denote these angles, and ι the vertical angle of the prism, there is $\text{ARR}' = \frac{\pi}{2} - \psi$, $\text{AR}'\text{R} = \frac{\pi}{2} - \psi'$.

Accordingly there is $\iota + \left(\frac{\pi}{2} - \psi\right) + \left(\frac{\pi}{2} - \psi'\right) = \pi$, whence

$$\iota = \psi + \psi'.$$

When the angle $AR'R$, which the ray within the prism makes with one of the sides towards the vertex, is obtuse, there is

$$AR'R = \frac{\pi}{2} + \psi', \text{ and, consequently, } \varepsilon = \psi - \psi'.$$

This formula, however, is included in the preceding, if the *positive* values of the angles ψ and ψ' be estimated from the perpendicular *towards the vertex* of the prism; for, this being understood, it is obvious that ψ or ψ' becomes *negative*, when the corresponding angle ARR' or $AR'R$ is *obtuse*.

From this it follows that the greatest angle of a prism, which can transmit a ray of light, is double the angle of total reflexion. For it is evident that $\varepsilon = \psi + \psi'$ is greatest, when ψ and ψ' are so; that is, when each of them is equal to $\theta_{,}$, the angle whose sine $= \frac{1}{m}$. Hence the greatest angle of the prism which will transmit a single ray is

$$\varepsilon = 2\theta_{,}.$$

(123.) Let φ and φ' denote the angles which the incident and emergent rays make with the perpendiculars to the surface at the points of incidence and emergence, the positive values of these angles being measured from the perpendicular *towards the base* of the prism. Also, let ψ and ψ', as before, denote the angles, which the course of the ray within the prism makes with the same perpendiculars, estimated *towards the vertex*. Then μ being the index of refraction of the prism, we have the following equations:

$$\sin.\varphi = \mu.\sin.\psi, \qquad \sin.\varphi' = \mu.\sin.\psi',$$
$$\psi + \psi' = \varepsilon;$$

by means of which the direction of the emergent ray is determined, that of the incident ray being known.

When the surfaces are *parallel*, or $\varepsilon = 0$, there is

$$\psi + \psi' = 0, \quad \text{whence } \sin.\psi' = -\sin.\psi;$$
$$\therefore \sin.\varphi' = -\sin.\varphi, \quad \text{and } \varphi' = -\varphi;$$

consequently the incident and emergent rays are parallel.

When the ray is incident perpendicularly upon the first surface of the prism, the determination of the direction of the emergent ray is easy. For, since $\varphi = 0$, we have, by the first and third of these equations, $\psi = 0$, $\psi' = \iota$; wherefore, substituting in the second, there is

$$\sin.\varphi' = \mu.\sin\iota.$$

Now, if δ denote the deviation at the second surface, which is the total deviation of the ray in this case, there is

$$\varphi' = \psi' + \delta = \iota + \delta,$$

$$\therefore \sin.(\delta + \iota) = \mu.\sin.\iota;$$

an equation which enables us to determine μ, the index of refraction of the prism, by measuring the deviation of a ray incident perpendicularly upon its first surface. This method, however, is not so convenient in practice as that of Newton, which will be presently explained.

(124.) The total deviation of the refracted ray is equal to the sum* of the deviations at incidence and emergence. Wherefore, denoting it by δ, there is

$$\delta = (\varphi - \psi) + (\varphi' - \psi') = \varphi + \varphi' - \iota,$$

since $\psi + \psi' = \iota$.

When a ray of light passes nearly perpendicularly through a thin prism, the total deviation is constant, and bears an invariable ratio to the angle of the prism.

For, in this case, the angles of incidence and refraction, being very small, may, without sensible error, be considered proportional to their sines; wherefore the equations of the preceding article become

$$\varphi = \mu\psi, \qquad \varphi' = \mu\psi';$$

and, adding, $\varphi + \varphi' = \mu(\psi + \psi') = \mu\iota$. Whence

$$\delta = \varphi + \varphi' - \iota = (\mu - 1)\iota.$$

(125.) Let it be required to determine in what case the total deviation of a ray, in passing through a prism of any refracting angle, is a *minimum*.

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* It is scarcely necessary to remark that the word *sum* is used here in the algebraic sense, and becomes the *difference* when one of the partial deviations is negative, or towards the vertex.

By the condition of the question there is

$$d.\delta = d(\phi + \phi') = 0.$$

Also, since $\psi + \psi' = \epsilon, \qquad d(\psi + \psi') = 0.$

If, then, we differentiate the equations

$$\sin.\phi = \mu.\sin.\psi, \qquad \sin.\phi' = \mu.\sin.\psi',$$

and substitute for $d\phi'$ and $d\psi'$ their values, $-d\phi$ and $-d\psi$, furnished by the equations first obtained, there is

$$\cos.\phi.d\phi = \mu.\cos.\psi.d\psi, \qquad \cos.\phi'.d\phi = \mu.\cos.\psi'.d\psi;$$

and, dividing the former by the latter,

$$\frac{\cos.\phi}{\cos.\phi'} = \frac{\cos.\psi}{\cos.\psi'}.$$

Now, squaring this equation, it may be written

$$\frac{1 - \sin.^2\phi}{1 - \sin.^2\phi'} = \frac{1 - \sin.^2\psi}{1 - \sin.^2\psi'}.$$

And, finally, substituting for $\sin.\phi$ and $\sin.\phi'$ their values, $\mu.\sin.\psi$, $\mu.\sin.\psi'$, and taking away the denominators, we obtain

$$\sin.^2\psi = \sin.^2\psi'.$$

Hence $\sin.\psi' = \pm \sin.\psi$, and $\therefore \psi' = \pm \psi$.

The lower sign cannot be employed, inasmuch as it would give $\epsilon = \psi + \psi' = 0$. There is, therefore,

$$\psi = \psi' = \frac{\epsilon}{2}.$$

Since ψ and ψ' are equal, it is evident that ϕ and ϕ' are also equal, and therefore the deviation is a minimum when the angles of incidence and emergence are equal.

The magnitude of the least deviation is determined by the equations

$$\delta = 2\phi - \epsilon,$$

$$\sin.\phi = \mu.\sin.\frac{\epsilon}{2}.$$

What has been said here of a single ray is evidently true of a pencil of parallel rays. If, therefore, a beam of solar light be refracted by a prism, and received upon a screen, the height of the depicted image will be a minimum when the refractions are equal at incidence and emergence. For the total deviation is equal to the sum of the angles which the incident and emergent rays make with the horizon; but the inclination of the incident ray to the horizon, or the altitude of the sun, is constant; therefore the inclination of the emergent ray to the horizon, and consequently the height of the image, will be least when the deviation is a minimum.

(126.) Newton's method of determining the index of refraction of any substance is derived from the principles just explained, and is as follows: The substance to be examined is formed into a prism, which is exposed to a beam of solar light, and so placed that the refractions shall be equal at incidence and emergence. It is easy to place the prism in this position; for we have only to move it slowly round its axis, and stop it at the point at which the solar image, between its descent and ascent, appears stationary; and, since the deviation is then a minimum, any small derangement from this position will cause no sensible change in the total deviation of the ray. The prism being fixed in this position, the sun's altitude and the inclination of the emergent beam to the horizon are then measured: their sum is equal to the total deviation. Wherefore, if the angle of the prism be ascertained, we have the angles of incidence and refraction at the entrance or emergence of the beam, and therefore the index of refraction. For, by the preceding article, there is

$$\psi = \frac{\varepsilon}{2}, \quad \varphi = \frac{\delta + \varepsilon}{2};$$

$$\therefore \mu = \frac{\sin. \frac{1}{2}(\delta + \varepsilon)}{\sin. \frac{1}{2}\varepsilon}.$$

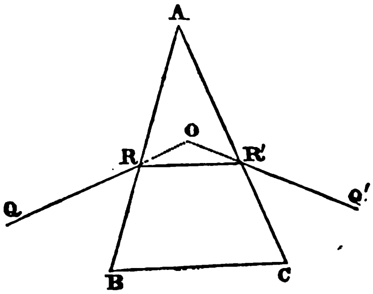
To determine the index of refraction of a fluid, it is only necessary to enclose the fluid in a hollow prism, whose sides are glass plates terminated by parallel planes, and then to take the measure as for a solid. For the direction of the emergent ray will be the same as if the ray had entered *immediately* from air into the fluid, and emerged in the same manner, without the interposition of the glass (116.).

M. Biot's contrivance is to bore a hole through the sides of a solid glass prism, communicating with another hole in the base through which the fluid is to be introduced; the apertures in the sides are then to be closed by glass plates.

(127.) We now proceed to establish directly a relation between the angles of incidence and emergence.

It is evident that this is to be accomplished by eliminating ψ and ψ' from the three equations (123.): the resulting equation will contain φ, φ', and ε. It will be found convenient, however, to employ, instead of the angles which the ray makes with the perpendiculars to the surfaces, the angles which it contains with the surfaces themselves.

Let α and α' denote the angles QRB, Q'R'C, which the incident and emergent rays form with the sides of the prism *towards the base;* β and β' the opposite angles, ARR', AR'R, which the course of the ray within the prism forms with those sides, *towards the vertex.* These are, respectively, the complements of the angles of incidence and refraction out of the surrounding medium into the prism, and we have, therefore, the following equations:



$$\cos.\alpha = \mu.\cos.\beta, \qquad \cos.\alpha' = \mu.\cos.\beta',$$

$$\beta + \beta' + \varepsilon = \pi;$$

which determine completely the direction of the emergent ray.

To eliminate β and β', we substitute in the second of these equations the value of β' given by the third, there is then

$$\cos.\alpha' = \mu.\cos.\left[\pi - (\beta + \varepsilon)\right] = -\mu.\cos.(\beta + \varepsilon)$$

$$= -\mu\left\{\cos.\beta.\cos.\varepsilon - \sin.\beta.\sin.\varepsilon\right\};$$

or, substituting for $\mu.\cos.\beta$ its value, $\cos.\alpha$, given by the first equation, and bringing that term to the other side,

$$\cos.\alpha' + \cos.\varepsilon.\cos.\alpha = \mu.\sin.\varepsilon.\sin.\beta.$$

Again, the first equation, multiplied by sin. ε, gives

$$\sin.\ \varepsilon\ .\ \cos.\ \alpha = \mu\ .\ \sin.\ \varepsilon\ .\ \cos.\ \beta.$$

And squaring and adding these two equations, we obtain

$$\cos.^2 \alpha + 2 \cos.\ \varepsilon\ .\ \cos.\ \alpha\ .\ \cos.\ \alpha' + \cos.^2 \alpha' = \mu^2.\ \sin.^2 \varepsilon,$$

an equation containing α, α', and ε only.

This equation may be put under a remarkable form; for, substituting for cos.$^2 \alpha$, cos.$^2 \alpha'$, their values, $\frac{1}{2}(1 + \cos.\ 2\alpha)$, $\frac{1}{2}(1 + \cos.\ 2\alpha')$, there is

$$\cos.^2 \alpha + \cos.^2 \alpha' = 1 + \tfrac{1}{2}(\cos.\ 2\alpha + \cos.\ 2\alpha')$$
$$= 1 + \cos.\ (\alpha + \alpha') \cos.\ (\alpha - \alpha').$$

Also, $\quad 2 \cos.\ \alpha\ .\ \cos.\ \alpha' = \cos.\ (\alpha + \alpha') + \cos.\ (\alpha - \alpha').$

Wherefore, making these substitutions, and subtracting sin.$^2 \varepsilon$ from both sides of the equation,

$$\mu^2.\ \sin.^2 \varepsilon - \sin.^2 \varepsilon =$$
$$\cos.^2 \varepsilon + \cos.\ \varepsilon . \big[\cos.(\alpha + \alpha') + \cos.(\alpha - \alpha')\big] + \cos.(\alpha + \alpha') \cos.(\alpha - \alpha'),$$

which is equivalent to

$$(\mu^2 - 1) \sin.^2 \varepsilon = \big[\cos.\ \varepsilon + \cos.\ (\alpha + \alpha')\big]\big[\cos.\ \varepsilon + \cos.\ (\alpha - \alpha')\big].$$

The angle contained by the incident and emergent ray

$$\text{QOQ}' = \text{A} + \text{ARO} + \text{AR}'\text{O} = \alpha + \alpha' + \varepsilon.$$

This angle will be a *maximum*, when $\alpha + \alpha'$ is so. Now, the first member of the above equation being constant, the two factors of the second vary reciprocally. Hence, when $\alpha + \alpha'$ is greatest, and therefore its cosine least, the first factor is least, and therefore the second greatest: it follows therefore that cos. $(\alpha - \alpha')$ will be the greatest possible in this case; it is therefore equal to unity, and $\alpha - \alpha' = 0$; *i. e.* the incident and emergent rays make equal angles with the sides of the prism.

(128.) The equation of the preceding article may be also employed to determine the index of refraction μ, when the angles α and α', which the incident and emergent rays make with the sides of the prism, are known. For this purpose it may be put under another form, better suited to logarithmic calculation; for, since the sum of two cosines is equal to twice the cosine of half the sum, multiplied by the cosine of half the difference, there is,

$$(\mu^2 - 1) \sin.^2 \varepsilon =$$

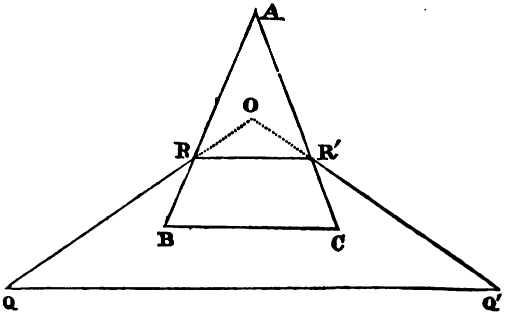
$$4 \cos. \frac{\alpha + \alpha' + \varepsilon}{2} \cos. \frac{\alpha + \alpha' - \varepsilon}{2} \cos. \frac{\varepsilon + \alpha' - \alpha}{2} \cos. \frac{\varepsilon + \alpha - \alpha'}{2};$$

or, making $\frac{\varepsilon + \alpha + \alpha'}{2} = \gamma$,

$$(\mu^2 - 1) \sin.^2 \varepsilon = 4 \cos. \gamma . \cos. (\gamma - \varepsilon) . \cos. (\gamma - \alpha) . \cos. (\gamma - \alpha'),$$

from which $\mu^2 - 1$, and therefore μ, is readily determined when ε, α, and α' are known.

The manner in which this is applied is the following:



The substance whose index of refraction is sought is to be formed into a prism, and any object, as Q, observed by the eye at Q', both directly and by means of the refracted ray, QRR'Q'. The angles at Q and Q' are then measured, as also the angle QRB $= \alpha$, which the incident ray makes with the side of the prism. Then, in the triangle QOQ', $O + Q + Q' = \pi$; but the angle O, contained by the incident and emergent rays, $= \alpha + \alpha' + \varepsilon$; wherefore, denoting the observed angles Q and Q' by η and θ, there is

$$\alpha + \alpha' + \varepsilon + \eta + \theta = \pi;$$

from which α' is determined, all the rest being known. Hence α, α', and ε being known, μ is determined by the equation above given*.

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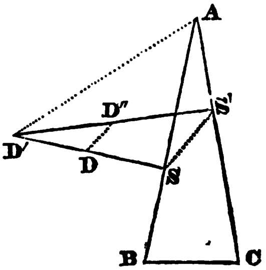
* For the results obtained by these and other methods, see Appendix.

(129.) A small pencil of rays being incident nearly perpendicularly upon a thin prism, it is required to find the focus of the refracted pencil.



Let D be the focus of the incident pencil, and DS a perpendicular from it upon the first surface; then, taking D′S $= \mu$.DS, D′ will be the focus after refraction by the first surface, and therefore the focus of the pencil incident on the second. Wherefore, drawing from D′ the perpendicular D′S′ on the second surface, and making D′S′ $= \mu$.D″S′, D″ will be the focus of the pencil after the second refraction.

It is evident that the line DD″, joining the foci of the incident and emergent pencils, is parallel to the line SS′, joining the intersections of the perpendiculars with the two surfaces.

It will not be difficult to calculate the position of the point D″ with respect to one of the sides of the prism, that of D being given.

For, referring the radiant to the first surface, the distances AS and DS are given; wherefore, denoting them by a and b, D′S $= \mu b$, is also given.

Again, referring the conjugate focus D″ to the second surface, and denoting the distances AS′ and D″S′ by α and β, there is D′S′ $= \mu\beta$.

Now, if we denote the distance D′A by r, the angle D′AS by θ, and the angle of the prism by ι, we have in the triangle D′S′A,

$$\alpha = r \cos.(\theta + \iota) = r(\cos.\theta.\cos.\iota - \sin.\theta.\sin.\iota),$$

$$\mu\beta = r \sin.(\theta + \iota) = r(\sin.\theta.\cos.\iota + \cos.\theta.\sin.\iota);$$

but in the triangle D′AS there is

$$a = r \cos.\theta, \quad \mu b = r \sin.\theta.$$

Wherefore, substituting these values in the expressions for α and β, we find

$$\alpha = a.\cos.\iota - \mu b.\sin.\iota,$$

$$\mu\beta = a.\sin.\iota + \mu b.\cos.\iota;$$

by which the position of the point D″ is completely determined.

Since the angle of the prism is supposed to be very small, we may, without much error, take cos. $\iota = 1$, sin. $\iota = \iota$, and we find

$$\alpha - a = -\mu\iota b,$$

$$\mu(\beta - b) = \iota a.$$

IV.

Of Images produced by Refraction at Plane Surfaces.

(130.) It has been already shown (112.) that when a pencil of rays, diverging from a point, is incident nearly perpendicularly upon a plane refracting surface, the rays will diverge, after refraction, from a point whose distance from the surface is to the distance of the focus of incident rays from the same in the constant ratio of the sine of incidence to the sine of refraction; or, that if δ and δ' denote the distances of the radiant and its conjugate from the surface, the relation between them is expressed by the equation

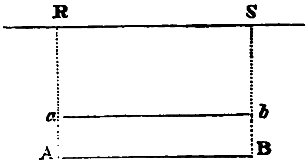
$$\delta' = m\delta.$$

Now, considering the image as the aggregate of the foci of the several pencils of refracted rays, which are incident from the several points of the object, its figure and position are found by letting fall perpendiculars from each point of the object on the refracting surface, and increasing or diminishing these ordinates in the constant ratio of $m : 1$; the extremities of the new ordinates determine the several points of the image.

It is evident from what has been said, that the figure of the image (if viewed perpendicularly to the refracting surface) is *similar* to that of the object, since the ordinates are in a constant ratio. But the position and magnitude will, in general, be different.

When the refraction is out of the rarer into the denser medium, *i. e.* when the object is in the rarer medium, the image is farther from the refracting surface than the object; and, on the contrary, it is nearer to the surface than the object, when the latter is in the denser medium, the perpendicular distances of the corresponding points of the object and image from the surface being in the constant ratio of 1 : *m*. Hence we see the reason why a lake or any place covered with water appears shallower than it really is; the image of the bottom being brought nearer to the surface by the refraction.

(131.) When the object is a *plane*, or its section rectilinear, if this plane be *parallel* to the surface, its image will be also parallel to it, at a distance greater or less in the ratio of *m* : 1; and the magnitude of the image will be the same as that of the object.

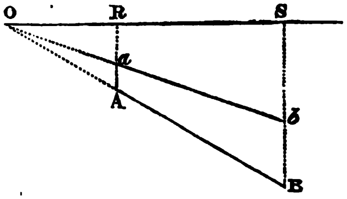


For AB being the object, and *ab* its image, we have

$$\frac{aR}{AR} = m = \frac{bS}{BS}.$$

And since AR = BS, there is also *a*R = *b*S, and therefore *ab* and RS are parallel; and moreover, since A*ab*B is a parallelogram, *ab* = AB.

If the plane of the object be *inclined* to the surface, that of the image will also be inclined to the same; and the tangents of the angles which the planes of the image and object make with the surface are to one another in the ratio of *m* : 1.



For AB being the object, *ab* the image, as before,

$$\frac{aR}{AR} = m = \frac{bS}{BS}, \quad \text{or} \quad \frac{bS}{aR} = \frac{BS}{AR};$$

from which it follows that the lines AB, *ab*, produced will intersect the surface at the same point, O; wherefore

$$\text{tang. } bOS : \text{tang. } BOS :: bS : BS :: m : 1.$$

Hence appears the reason why a stick, or any straight object, partly immersed in water, in a direction inclined to the surface, appears broken; the image of the immersed part forming a less angle with the surface than the stick itself, and therefore appearing inclined to the part which is not immersed.

The magnitudes of the object and image are evidently to one another as the secants of the angles at which they are inclined to the surface; for

$$\text{BA} : ba :: \text{AO} : a\text{O} :: \text{sec. AOR} : \text{sec. } a\text{OR}.$$

(132.) Let the object be presented to a refracting medium bounded by two parallel plane surfaces.

It has been shown (119.) that when a pencil of rays, diverging from a point, is incident nearly perpendicularly on such a medium, they will diverge, after refraction, from a point nearer the surface if it be denser than the surrounding medium, farther from it if rarer; the interval between the foci of the incident and refracted rays bearing a constant ratio to the thickness of the medium. Hence, when a luminous object is presented to such a medium, since the perpendicular distances of each point of the object from the surface are increased or diminished by the *same* quantity, the image will be *parallel* to the object, and its *figure* and *magnitude* the same; the only difference being in the *position* of the image, which is nearer to, or farther from the surface than the object, according as the medium is denser or rarer than that which surrounds it.

(133.) Let the object be presented to a prism whose refracting angle is very small.

It will be readily seen, from what has been said of the image produced by a single surface, that, if the section of the object be a right line, that of the image will also be a right line, inclined to the surface at an angle which may be readily calculated. For, if θ, θ', θ'', denote the angles made by the object and its images after the first and second refraction with the first surface of the prism, it will appear from (131.) that

$$\tan.\theta' = \mu \tan.\theta, \qquad \tan.(\theta' + \varepsilon) = \mu \tan.(\theta'' + \varepsilon);$$

in which ε denotes the angle of the prism. And, if θ' be eliminated from these equations, θ'', the inclination of the image after refraction by both surfaces, will be expressed in terms of θ and ε.

But, in general, whatever be the form of the object, the figure of the image may be calculated from the formulæ which determine the conjugate foci in this case. For it has been shown (129.) that if a and b denote the coordinates of the focus of the incident pencil, referred to the first surface of the prism, α and β, those of the focus of the emergent pencil, referred to the second surface, the relation between them is expressed by the equations:

$$\alpha - a = -\mu\epsilon b,$$

$$\beta - b = \frac{1}{\mu}\epsilon a.$$

Now, if the object be of any regular figure, the relation between the coordinates, a and b, is given. Wherefore, if a and b be eliminated, by means of the two equations just given, combined with the equation of the object itself referred to the first surface of the prism, the result will be an equation containing α and β only, which will therefore determine the form of the image.

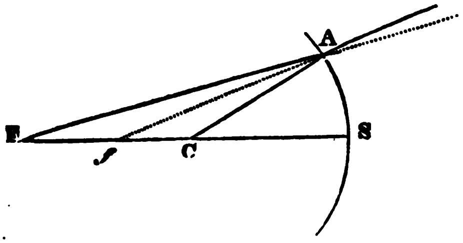
CHAPTER VI.

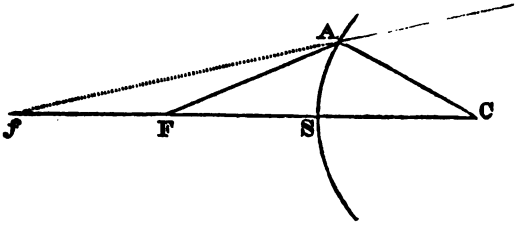
OF LIGHT REFRACTED BY SPHERICAL SURFACES.

I.

Refraction of a small Pencil of Rays incident perpendicularly upon a single spherical Surface.

(134.) GIVEN the direction of a ray of light incident on a spherical surface, bounding two media of different densities; it is required to find the direction of the refracted ray.





Let FA be the incident ray; AS the section of the spherical surface formed by the plane containing the incident ray and the centre; and CS any diameter of this section meeting the incident ray in F. Also, let Af be the refracted ray, or the refracted ray produced, meeting this diameter in f; and let CA be the radius drawn to the point of incidence.

Then, in the triangles FAC, fAC, there is

$$\mathrm{FC} = \mathrm{FA}\,.\frac{\sin.\mathrm{FAC}}{\sin.\mathrm{FCA}}, \qquad f\mathrm{C} = f\mathrm{A}\,.\frac{\sin.f\mathrm{AC}}{\sin.f\mathrm{CA}};$$

and if we divide the former of these equations by the latter, and observe that $\frac{\sin.\mathrm{FAC}}{\sin.f\mathrm{AC}} = \frac{\sin.\mathrm{inc.}}{\sin.\mathrm{ref.}} = m$, we have

$$\frac{\mathrm{FC}}{f\mathrm{C}} = m.\frac{\mathrm{FA}}{f\mathrm{A}}.$$

From this equation, by substituting for FA and fA their values, expressed in terms of FC and fC, the radius of the spherical surface, and the angle at the centre, we may obtain a general relation between the distances FC and fC, whatever be the incidence. This investigation will be found in a subsequent part of the chapter.

(135.) A small pencil of rays, diverging from or converging to a point, being incident perpendicularly upon a spherical surface, it is required to find the focus of the refracted pencil.

Let F be the focus of the incident pencil; FC the line connecting it with the centre, and meeting the surface in S: this line, being perpendicular to the refracting surface, is the axis of the pencil. Now, FA being any ray of the incident pencil, the point f, in which the refracted ray meets the axis, is determined by the equation $\frac{\mathrm{FC}}{f\mathrm{C}} = m\,.\frac{\mathrm{FA}}{f\mathrm{A}}$. But the incident pencil being very small, and perpendicular to the refracting surface, the rays which compose it are nearly coincident with the axis; hence the point A coincides nearly with S, and FA and fA become FS and fS, respectively. Accordingly, the ultimate position of the point in which the refracted rays meet the axis, when the breadth of the incident pencil is indefinitely diminished, is determined by the equation,

$$\frac{\mathrm{FC}}{f\mathrm{C}} = m.\frac{\mathrm{FS}}{f\mathrm{S}};$$

from which we learn that the distances of the conjugate foci from the centre are to one another in a ratio compounded of the ratio of their distances from the surface, and of the ratio of the sines of incidence and refraction.

(136.) Let FS and *f*S, the distances of the conjugate foci from the surface, be denoted by δ and δ', and the radius CS by r, then, the refracting surface being *concave*, FC $= \delta - r$, *f*C $= \delta' - r$, and the equation of the preceding article is written

$$\frac{\delta - r}{\delta' - r} = m . \frac{\delta}{\delta'};$$

or, taking away the denominators, and dividing the result by $r\delta\delta'$, there is

$$m\left(\frac{1}{\delta'} - \frac{1}{r}\right) = \frac{1}{\delta} - \frac{1}{r};$$

or, if we denote the reciprocals of r, δ and δ', by ϱ, α and α', as before (54.),

$$m(\alpha' - \varrho) = \alpha - \varrho.$$

(137.) When the refracting surface is *convex*, we have, from the second of the preceding figures, FC $= \delta + r$, *f*C $= \delta' + r$; for this case, therefore, the equation of (135.) becomes

$$\frac{\delta + r}{\delta' + r} = m . \frac{\delta}{\delta'};$$

from which we obtain, using the same notation as before,

$$m(\alpha' + \varrho) = \alpha + \varrho,$$

a result differing from that of the preceding article in the sign of ϱ only.

Now, if we observe that the radius r, in this case, is measured from the surface in a direction opposite to that in the former, it will be evident that the formula

$$m(\alpha' - \varrho) = \alpha - \varrho$$

really includes all cases, if we only agree to consider the distances r, δ, and δ' (and therefore their reciprocals, ϱ, α, and α') as *positive*, when measured from the surface *towards the incident light; negative*, when in the opposite direction. This, it is obvious, is the same as assuming the positive values of the quantities ϱ, α, and α' to belong to the case of a *concave* surface, and *divergent* rays.

When the refracting surface is plane, $\varrho = 0$, and the general formula becomes

$$m\alpha' = \alpha, \quad \text{or} \quad \delta' = m\delta,$$

as we have already obtained (112.).

(138.) The preceding formula includes also the case of reflexion. For, in reflected light, since the angles of incidence and reflexion are equal, and lie at opposite sides of the normal, their sines are equal with opposite signs, and therefore, in this case, $m = -1$. If, then, this particular value of m be substituted in the general formula (136.), it becomes

$$\alpha + \alpha' = 2\varrho,$$

agreeing with the result already obtained*.

(139.) The formula of (136.), as it has been found to include the case in which the light is incident upon the convex surface, as well as that in which it is incident upon the concave; so it is also true whether the light passes from the rarer into the denser medium, or in the contrary direction; m denoting in all cases the ratio of the sines of incidence and refraction. The value of m, however, is different in these two cases, being equal to μ in the former case, and to $\frac{1}{\mu}$ in the latter; μ denoting the ratio of the sines of the angles which the portions of the ray in the rarer and denser medium, respectively, make with the perpendicular to the surface at the point of incidence. To distinguish between the cases, accordingly, we have only to substitute these values for m in the general equation (136.), and when the light is incident from the rarer into the denser medium, we have

$$\mu(\alpha' - \varrho) = \alpha - \varrho;$$

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* In this manner the whole theory of reflected light might be derived from that of refracted light, by simply making $m = -1$ in the results of the latter theory. This method, however, is not so well adapted to the purposes of the student as that which has been employed.

when from the denser into the rarer,

$$\alpha' - \varrho = \mu(\alpha - \varrho).$$

It is evident that one of these formulæ is obtained from the other by transposing α and α'; from which we conclude that the foci of the incident and refracted pencils are convertible, or that, if the light be incident in the opposite direction, diverging from, or converging to, the focus of the refracted rays, it will be refracted diverging from, or converging to, the focus of the former incident pencil.

(140.) When the incident rays are parallel, $\alpha = 0$; and, if the resulting value of α' be denoted by φ, there is

$$\varphi = \frac{m-1}{m}\varrho.$$

This quantity φ, or the *vergency* given to parallel rays by refraction at any surface, is called the *power* of that surface; we learn, therefore, that the power of any surface varies directly as its *curvature.*

This result may be otherwise expressed: for, if we substitute for φ and ϱ their values, $\frac{1}{f}$ and $\frac{1}{r}$, f denoting the distance of the principal focus from the surface, we have

$$f = \frac{mr}{m-1}.$$

This formula may be put under the form

$$m(f - r) = f;$$

from which it appears that the distance of the principal focus from the surface is to its distance from the centre in the constant ratio of the sine of incidence to the sine of refraction.

(141.) When the rays are incident from the rarer into the denser medium, $m = \mu$, and the equation of the preceding article becomes

$$\varphi = \frac{\mu - 1}{\mu}\varrho.$$

Wherefore, μ being always greater than unity, the sign of φ is the same with that of ϱ. Accordingly, when ϱ is *positive*, or the refracting surface *concave*, the refracted rays *diverge* from the principal focus; and, on the contrary, they *converge* to the same, when ϱ is *negative*, or the refracting surface *convex* (137.).

When the rays proceed from the denser into the rarer medium, $m = \frac{1}{\mu}$, and substituting in the equation of the preceding article,

$$\varphi = -(\mu - 1)\varrho.$$

In this case, then, the sign of φ is opposite to that of ϱ, and it follows therefore, from the rule of signs laid down (137.), that a *concave* refracting surface of a rarer medium will give a *convergence* to parallel rays; and a *convex*, *divergence*.

It appears from the preceding, that any spherical surface, bounding two media, has two principal foci, one of rays proceeding from the rarer into the denser medium, and the other of those which proceed in the opposite direction.

If the curvature of the surface and its power, in the latter case, be denoted by ϱ' and φ', to distinguish them from the former; there is $\varrho' = -\varrho$, since the curvature of the surface, in the two cases, lies in opposite directions with respect to the incident light; wherefore, from the preceding formulæ,

$$\varphi' = \mu\varphi.$$

From which we conclude that the two foci are similarly situated with respect to the incident light, and therefore lie at opposite sides of the surface; and that their distances from the surface are in the constant ratio of the sines of incidence and refraction.

(138.) Returning to the general equation (136.), which is equivalent to

$$m\alpha' = \alpha + (m - 1)\varrho,$$

it is evident that, when the term $(m - 1)\varrho$ is *positive*, *i. e.* when the light is incident upon the *concave* surface of a *denser*, or the convex of a *rarer* medium, α' will be always *positive*, unless when α is negative and greater than $(m - 1)\varrho$. In these cases, therefore, the refracted rays always *diverge*, unless when the incident rays converge to some point nearer to the surface than the principal focus of rays coming in the opposite direction.

On the contrary, when $(m - 1)\varrho$ is *negative*, or the rays incident upon the *convex* surface of a *denser*, or the *concave* of a *rarer* medium, α' will be *negative*, unless when α is positive and greater than $(m - 1)\varrho$; and accordingly the refracted rays, in these cases, always *converge*, unless when the incident rays diverge from some point nearer to the surface than the principal focus of rays proceeding in the opposite direction.

(143.) If the preceding equation be differentiated, we find

$$m d\alpha' = d\alpha, \quad \text{or} \quad m d\delta' = \frac{\delta'^2}{\delta^2} d\delta;$$

from which we learn that the conjugate foci move always in the *same direction*, the increments of their distances from the surface having always the same sign. This is the contrary of that which takes place in reflexion.

It will be easy to see the corresponding positions of the foci.

1. When $\alpha = 0$, $\alpha' = \frac{m-1}{m} \varrho = \varphi$. Therefore, when the radiant, or the focus of incident rays, is infinitely distant, its conjugate arrives at the principal focus.

2. When $\alpha = -(m-1)\varrho$, $\alpha' = 0$; *i. e.* when the radiant coincides with the principal focus of rays coming in the opposite direction, its conjugate moves off to an infinite distance.

3. When $\alpha = \varrho$, $\alpha' = \varrho$; and, accordingly, when the radiant coincides with the centre of the surface, its conjugate meets it at the same point.

4. When α is infinite, α' becomes also infinite; and therefore when the radiant coincides with the surface, its conjugate meets it there.

Hence the march of the two foci may be easily traced.

(144.) In the preceding article we have seen that the two foci always move in the same direction, and coincide at the surface and the centre. In order to determine when the interval between them is a maximum, we must have $d\delta' = d\delta$, or $\frac{d\alpha'}{\alpha'^2} = \frac{d\alpha}{\alpha^2}$. But $m d\alpha' = d\alpha$; and dividing this by the equation of condition just found, in order to eliminate $d\alpha'$ and $d\alpha$, there is

$$m\alpha'^2 = \alpha^2.$$

Now, if we multiply this result by m, and extract the square root, there is $ma' = \alpha\sqrt{m}$; and, substituting this in the equation $ma' = \alpha + (m - 1)\varrho$, we obtain

$$\alpha(\sqrt{m} - 1) = (m - 1)\varrho, \quad \text{or } \alpha = (\sqrt{m} + 1)\varrho.$$

(145.) It is sometimes convenient to compute the distances *from the centre*, instead of the surface. This is readily done; for if the distances FC and fC (see figures, page 101) be denoted by d and d', when the refracting surface is *convex*, there is FS $= d - r$, fS $= d' - r$; and substituting in equation (135.),

$$\frac{d}{d'} = m \cdot \frac{d - r}{d' - r};$$

and, if we take away the denominators, divide the result by rdd', and denote the reciprocals of r, d and d', by ϱ, u and u', we obtain

$$u' - \varrho = m(u - \varrho).$$

When the refracting surface is *concave*, there is FS $= d + r$, fS $= d' + r$; whence there is, in this case,

$$\frac{d}{d'} = m \cdot \frac{d + r}{d' + r}, \quad \text{whence } u' + \varrho = m(u + \varrho),$$

a result differing from the former in the sign of r or ϱ only.

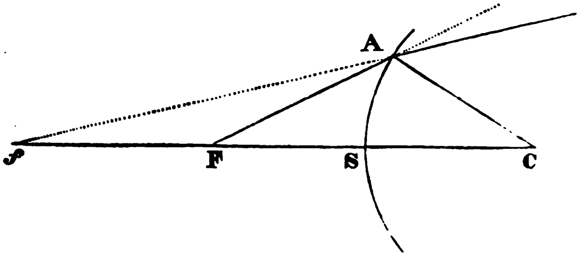
In this case, however, it is evident that the radius is measured from the centre in an opposite direction, with respect to the incident light, to that in the former; if, then, the positive values of the distances d, d', and r, be estimated from the centre *towards the incident light*, in conformity with the arbitrary convention respecting signs already laid down, in this latter case the radius r, or its reciprocal ϱ, must be considered as negative, and the formula

$$u' - \varrho = m(u - \varrho)$$

will embrace all cases.

II.

Aberration in Refraction at a single spherical Surface.



(146.) In the preceding section we have sought only the *ultimate* position of the point, *f*, in which the refracted ray meets the axis, when the arc SA is diminished *ad infinitum.* It is readily seen that the number of the refracted rays collected at that point will be far greater than at any other, and that therefore that point is to be considered as the virtual focus of the refracted pencil. As, however, each ray of the refracted pencil *actually* meets the axis in a different point, it is important to determine the space through which they are diffused along the axis, when the arc SA is finite. This is evidently obtained by seeking the distance of the intersection of the extreme ray with the axis from the surface, or centre; the difference between this and its ultimate value, when $SA = 0$, will be the *space of diffusion*, or, as it is more frequently called, the *aberration* of the extreme ray.

It has been already proved (134.) that whatever be the aperture, there exists the relation

$$\frac{FC}{fC} = m.\frac{FA}{fA}.$$

Now, if the distances from the centre, FC and fC, be denoted by d and d'; the radius CA by r; and the angle CAS by θ; there is

$$FA^2 = d^2 + r^2 - 2dr\cos.\theta, \quad fA^2 = d'^2 + r^2 - 2d'r\cos.\theta.$$

Wherefore, squaring the preceding equation, and substituting these values,

$$\frac{d^2}{d'^2} = m^2 . \frac{d^2 + r^2 - 2dr \cos . \theta}{d'^2 + r^2 - 2d'r \cos . \theta};$$

or,

$$\frac{r^2}{d'^2} - 2\frac{r}{d'} \cos . \theta + 1 = m^2 \left(\frac{r^2}{d^2} - 2\frac{r}{d} \cos . \theta + 1 \right),$$

or, dividing both sides by r^2, and denoting by u, u', and ϱ, the reciprocals of d, d', and r, we have

$$u'^2 - 2u'\varrho \cos . \theta + \varrho^2 = m^2 [u^2 - 2u\varrho \cos . \theta + \varrho^2];$$

an equation which gives the relation between u and u', the reciprocals of the distances of the intersections of the incident and refracted rays with the axis, whatever be the value of θ, the angle at the centre.

If we substitute $\varrho^2 (\cos .^2 \theta + \sin .^2 \theta)$ for ϱ^2, on both sides of the equation, it will assume the form

$$(u' - \varrho \cos . \theta)^2 + \varrho^2 \sin .^2 \theta = m^2 \left[(u - \varrho \cos . \theta)^2 + \varrho^2 \sin .^2 \theta \right];$$

from which we have

$$u' = \varrho \cos . \theta \pm \sqrt{\left[m^2 (u - \varrho \cos . \theta)^2 + (m^2 - 1) \varrho^2 \sin .^2 \theta \right]},$$

the general value of u'.

When the incident ray is parallel to the axis, $u = 0$, and this expression becomes

$$u' = \varrho \left[\cos . \theta \pm \sqrt{(m^2 - \sin .^2 \theta)} \right].$$

The value of u' being a function of θ, the intersection of each ray of the refracted pencil with the axis is, in general, different. There is one case, however, in which the value of u' is independent of θ; it is that in which the coefficient of $\cos . \theta$, in the general equation, is nothing, or $u' = m^2 u$. If we substitute this value for u', in the same equation, we obtain

$$mu = \varrho, \quad \text{or} \quad d = mr.$$

When the distance of the radiant, then, has this value, all the rays will be refracted *exactly* to the same point, whatever be their incidence.

(147.) In order to obtain an approximate value of u', when the angle θ is small, we shall return to the general equation, which, if we substitute for cos. θ its value $1 - v$, (v denoting the versed sine of θ) assumes the following form:

$$(u' - \varrho)^2 + 2u'\varrho v = m^2\left[(u - \varrho)^2 + 2u\varrho v\right].$$

If we make $v = 0$ in this equation, and denote the resulting value of u' by $u_{\prime}$, there is

$$u_{\prime} - \varrho = m(u - \varrho);$$

an equation which agrees with that already found (145.), and which determines $u_{\prime}$ the ultimate value of u', when the incident ray coincides with the axis.

Again, when the angle θ is so small that all the powers of its versed sine, above the first, may be neglected, we have, by M'Laurin's theorem,

$$u' = u_{\prime} + \left(\frac{du'}{dv}\right)v;$$

$\left(\frac{du'}{dv}\right)$ denoting the value of $\frac{du'}{dv}$, when $v = 0$.

But, differentiating the preceding equation, we have

$$(u' - \varrho)\,du' + \varrho\,(vdu' + u'dv) = m^2u\varrho dv.$$

Whence, making $v = 0$, we obtain $\left(\frac{du'}{dv}\right) = \frac{m^2u - u_{\prime}}{u_{\prime} - \varrho}\varrho$, and therefore

$$u' - u_{\prime} = du' = \frac{m^2u - u_{\prime}}{u_{\prime} - \varrho}\varrho v.$$

Finally, if in this result we substitute for $u_{\prime}$ its value in u, given above, we obtain

$$du' = \frac{m-1}{m} \cdot \frac{mu + \varrho}{u - \varrho}\varrho v.$$

When the incident rays are parallel, $u = 0$, and the variation of u' becomes

$$du' = \frac{1-m}{m}\varrho v.$$

The value of du' in the case of reflected light may be obtained from the preceding, by making $m = -1$, in the general expression just found, which thus becomes

$$du' = -2\varrho v,$$

agreeing with the result obtained (61.).

(148.) In the preceding investigations the distances have been computed from the centre: it is in general more convenient, however, to measure them from the surface; and it is evident that the results in this case will be obtained from the former by a simple transformation. For, if δ and δ' denote the distances from the surface FS and *fs*, there is $d = \delta + r$, $d' = \delta' + r$: or, denoting the reciprocals of d and d', δ and δ', by u and u', α and α', respectively,

$$u = \frac{\alpha\varrho}{\alpha + \varrho}, \quad u' = \frac{\alpha'\varrho}{\alpha' + \varrho};$$

and these values being substituted in the equation which gives the ultimate value of u', we obtain

$$\alpha + \varrho = m(\alpha' + \varrho).$$

Making the same substitution in the expression for du', there is

$$du' = (\alpha' - m\alpha)v;$$

but, if we differentiate the value of u' expressed in terms of α', there is $du' = \frac{\varrho^2 . d\alpha'}{(\alpha' + \varrho)^2}$; whence

$$d\alpha' = \left(1 + \frac{\alpha'}{\varrho}\right)^2 du' = \left(1 + \frac{\alpha'}{\varrho}\right)^2 (\alpha' - m\alpha)v.$$

The preceding investigations have been adapted to the case in which the refracting surface is *convex* towards the incident light; but as, in this case, the radius r, measured from the surface, is *negative*, in order to accommodate the formulæ to the case in which the quantities entering them are positive, the sign of ϱ must be changed, and they become

$$\alpha - \varrho = m(\alpha' - \varrho),$$

$$d\alpha' = \left(1 - \frac{\alpha'}{\varrho}\right)^2 (\alpha' - m\alpha)v.$$

The former of which equations determines the *ultimate* value of α', when the arc, or its versed sine, is *evanescent*; the latter, the difference between this ultimate value and the *approximate* value of α', when v is finite, though small.

(149.) To express $d\alpha'$ in terms of the aperture, it is to be observed that, when the arc θ is small, its versed sine

$$v = \tfrac{1}{2}\sin.^2\theta,\ q.p = \tfrac{1}{4}\varrho^2 r^2 \sin.^2\theta = \tfrac{1}{4}\varrho^2 x^2.$$

x denoting the semiaperture; wherefore, substituting

$$d\alpha' = (\alpha' - \varrho)^2 (\alpha' - m\alpha).\frac{x^2}{2};$$

from which it appears that the variation of α' varies, cæt. par., as the square of the aperture.

If α' be eliminated from this expression by means of its value in α, it becomes

$$d\alpha' = \frac{m-1}{2m^3}(\varrho - \alpha)^2[\varrho - (m+1)\alpha]\,x^2.$$

When the incident rays are parallel, $\alpha = 0$, and $\alpha' = \varphi$, and this expression becomes

$$d\varphi = \frac{m-1}{2m^3}\varrho^3 x^2;$$

or, substituting for ϱ its value in φ, namely, $\frac{m\varphi}{m-1}$,

$$d\varphi = \frac{\varphi^3.x^2}{2(m-1)^2}.$$

(150.) The quantity $d\alpha'$ being found, the *aberration* is easily obtained; for δ' being the distance of the intersection of any ray with the axis measured from the surface,

$$\delta' = \frac{1}{\alpha'},\quad \text{whence}\ d\delta' = -\frac{d\alpha'}{\alpha'^2}.$$

The quantity $d\delta'$ is the aberration.

Thus, for parallel rays, $df = -\frac{d\varphi}{\varphi^2}$; whence we find

$$df = -\frac{\varphi.x^2}{2(m-1)^2} = \frac{-x^2}{2(m-1)^2 f};$$

from which it appears that the aberration from the principal focus varies as the square of the aperture directly and inversely as the focal length.

(151.) Returning to the general expression for $d\alpha'$, it is evident that there are two cases in which it becomes nothing, and in which, therefore, the aberration vanishes. These are when $\varrho = \alpha$, and when $\varrho = (m + 1)\alpha$. In the former case the incident rays diverge from, or converge to, the centre, and therefore undergo no refraction; and, in the latter, the distance of the focus of incident rays from the surface, or $\delta = (m + 1)r$, and all the rays are refracted accurately to the same point (146).

With respect to the sign of $d\alpha'$, it will evidently be determined by that of the quantity $\varrho - (m + 1)\alpha$, being the same with the sign of this quantity, when $m > 1$, or the rays incident from the rarer into the denser medium; and the opposite, when the light proceeds in the contrary direction.

Thus when *diverging* rays are incident upon the *concave* surface of a *denser* medium, ϱ and α are both *positive;* and, therefore, $d\alpha'$ will be positive, negative, or nothing, according as $\varrho > (m + 1)\alpha$, $\varrho < (m + 1)\alpha$, or $\varrho = (m + 1)\alpha$; *i. e.* according as δ is greater, less than, or equal to $(m + 1)r$. When *converging* rays are incident upon the concave surface of a denser medium, α is negative, and the value of $d\alpha'$ is, in all cases, positive.

When the rays are incident upon the *convex* surface of a denser medium, ϱ is *negative;* and it is easily seen that, in the case of *diverging* rays, the value of $d\alpha'$ is always *negative;* while, for converging rays, it will be negative, positive, or nothing, according as $\varrho > (m + 1)\alpha$, $\varrho < (m + 1)\alpha$, or $\varrho = (m + 1)\alpha$.

Finally, when the light passes *from the denser* into the rarer medium, the sign of $d\alpha'$ is the opposite to that which it has when the light proceeds in the contrary direction, other circumstances being the same. The sign of the aberration is always the opposite to that of $d\alpha'$ (150.).

In the case of parallel rays, it is evident (150.) that the sign of the aberration df, is, in all cases, the opposite to that of f, the focal length. Hence it follows that the intersection of the extreme rays with the axis is *always nearer to the surface* than that of the central ones.

III.

Successive Refraction of a small Pencil of Rays incident perpendicularly upon several spherical Surfaces.

(152.) We may now proceed to consider the successive refraction of a small pencil of rays by several media, the surfaces bounding which are supposed to be spherical, and their centres disposed along the same right line. This right line is obviously perpendicular to all the surfaces, and is called the common axis of the spherical surfaces.

We shall, in the first place, suppose that the intervals between the successive surfaces are inconsiderable, so that the focal distance of the rays after refraction by any one of them is that of the rays incident on the next. If, then, m', m'', m''', &c. denote the indices of refraction at the 1st, 2d, 3d surfaces, &c. respectively; α the vergency of the incident pencil, and α', α'', α''', &c. the vergencies after refraction by the several surfaces, the relations which exist among these quantities are determined by the equations

$$m'\alpha' = \alpha + (m' - 1)\rho', \qquad m''\alpha'' = \alpha' + (m'' - 1)\rho'',$$
$$m'''\alpha''' = \alpha'' + (m''' - 1)\rho''', \ \&\text{c.}$$

Now, if we multiply the 2d of these equations by μ', the 3d by μ'', &c. (μ', μ'', μ''', &c. denoting the indices of refraction between the original medium and the 1st, 2d, 3d medium, &c. respectively) and observe that

$$m' = \mu', \qquad m''\mu' = \mu'', \qquad m'''\mu'' = \mu''', \ \&\text{c.}$$

they become

$$\begin{aligned} \mu'\alpha' &= \alpha + (\mu' - 1)\rho', \\ \mu''\alpha'' &= \mu'\alpha' + (\mu'' - \mu')\rho'', \\ \mu'''\alpha''' &= \mu''\alpha'' + (\mu''' - \mu'')\rho''', \\ \&\text{c.} &= \&\text{c.} \end{aligned}$$

and adding these equations together, and denoting the number of surfaces by n, there is

$$\mu^{(n)}\alpha^{(n)} = \alpha + (\mu' - 1)\rho' + (\mu'' - \mu')\rho'' + \&\text{c.} + (\mu^{(n)} - \mu^{(n-1)})\rho^{(n)};$$

an equation which determines $\alpha^{(n)}$, the vergency of the refracted pencil, when that of the incident pencil is given.

When the intervals between the successive surfaces are too considerable to be neglected, the focal distance of the rays after refraction by any one of them is no longer that of the rays incident upon the next, the difference between these distances being equal to the interval between the surfaces. The equations expressing these relations, together with a system of equations similar to the preceding, expressing the relation between the focal distances of the incident and refracted rays for each separate surface, will determine the problem. The elimination amongst these equations, however, leads to results of great complexity.

(153.) If φ denote the *power* of the system, or the value of $\alpha^{(n)}$, when the incident rays are parallel, or $\alpha = 0$, the preceding equation is resolvable into the following:

$$\mu^{(n)}\varphi = (\mu' - 1)\varrho' + (\mu'' - \mu')\varrho'' + \&c. + (\mu^{(n)} - \mu^{(n-1)})\varrho^{(n)},$$
$$\mu^{(n)}\alpha^{(n)} = \alpha + \mu^{(n)}\varphi.$$

When the ray re-emerges into the original medium, $\mu^{(n)} = 1$, and these equations become

$$\varphi = (\mu' - 1)\varrho' + (\mu'' - \mu')\varrho'' + \&c. + (1 - \mu^{(n-1)})\varrho^{(n)},$$
$$\alpha^{(n)} = \alpha + \varphi.$$

Thus, when a pencil of rays passes through a medium bounded by two spherical surfaces, and re-emerges into the original medium, the power of the system becomes

$$\varphi = (\mu' - 1)\varrho' + (1 - \mu')\varrho'' = (\mu' - 1)(\varrho' - \varrho'').$$

Such a combination is called a *lens*. As this case, however, is one of the greatest practical importance, it will merit a distinct consideration.

(154.) A *spherical lens* is a solid bounded by two spherical surfaces, or by a plane and a spherical surface. The *axis* of the lens is a right line passing through the centres of both surfaces.

There are six different forms of the spherical lens:—1. The *double convex*, in which both surfaces are convex externally. 2. The *double concave*, in which both are concave. 3. The *meniscus*, in which one surface is convex and the other concave, the curvature of the convex being the greater. 4. The *concavo-convex*, in which one surface is convex and the other concave, the curvature of the concave being the greater. 5. The *plano-convex*, one of whose surfaces is plane and the other convex. 6. The *plano-concave*, in which one surface is plane and the other concave.

These varieties are indicated algebraically by the signs of the radii, or of the curvatures of the surfaces. The curvatures of those surfaces being *positive* which are *concave* towards the incident light; *negative*, of those which are *convex*.

(155.) We shall, for the present, confine our attention to the case of *central* rays; those, namely, which diverge from, or converge to some point in the axis of the lens, and are nearly coincident with that axis. Such rays, it is obvious, are incident nearly perpendicularly upon both surfaces of the lens.

The focus of central rays, incident upon a lens, being given, it is required to find its conjugate, or the focus of the refracted rays.

Let α and β denote the *vergencies* of the incident and refracted rays, after refraction by the first surface; β' and α' the corresponding quantities for the second surface; ϱ and ϱ' the *curvatures* of the two surfaces; and μ the index of refraction in passing from the surrounding medium into that of which the lens is composed; then the relations amongst these quantities are given by the equations

$$\mu(\beta - \varrho) = \alpha - \varrho,$$
$$\mu(\beta' - \varrho') = \alpha' - \varrho'.$$

Now, if the distance between the surfaces, or the thickness of the lens, is so small that it may be neglected, since the focus of the pencil after refraction by the first surface is that of the pencil incident on the second,

$$\beta' = \beta.$$

Wherefore, subtracting the former of these equations from the latter, we have $\mu(\varrho - \varrho') = \alpha' - \alpha + \varrho - \varrho'$, or

$$\alpha' - \alpha = (\mu - 1)(\varrho - \varrho');$$

an equation which determines α', the vergency of the refracted rays, when α, the vergency of the incident rays, and the curvatures of the surfaces, are known.

This equation comprehends every case that may arise, if we remark only that the *positive* values of α and α' belong to the case of *divergent* rays; those of ϱ and ϱ', to that in which the surfaces are *concave* towards the incident light.

(156.) When the incident rays are parallel, $\alpha = 0$, and if we denote the resulting value of α' by φ, we have

$$\varphi = (\mu - 1)(\varrho - \varrho').$$

The quantity φ may be called the *absolute* vergency of the refracted rays, or the *power* of the lens; and the equation may be thus enunciated:—"The power of a lens is to the difference* of the curvatures of its surfaces in the constant ratio of the difference of the sines of incidence and refraction to the sine of refraction."

Substituting this value in the equation of the preceding article, it becomes

$$\alpha' = \alpha + \varphi;$$

which may be thus expressed:—"The excess of the vergency of the refracted above that of the incident rays is constant, and equal to the power of the lens."

(157.) From the value of φ, obtained in the preceding article, we learn that the power of the lens, or its reciprocal, the principal focal length, remains the same whichever side of the lens be turned towards the incident light. For, if the lens be turned round, ϱ becomes ϱ' and *v. v.*; but, moreover, the signs of both are changed, the opposite surface being now offered to the incident light; wherefore the value of φ remains the same, both in sign and magnitude, as before.

Accordingly, in the different cases which we proceed to examine, we are at liberty to suppose $\varrho > \varrho'$, or the surface of greater curvature, to be turned towards the incident light, inasmuch as this supposition will not affect the result.

In the *concavo-convex* lens, therefore, ϱ and ϱ' being both positive, the value of φ is

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* The word *difference* is here taken in the algebraic sense, and becomes the sum when the curvatures are of opposite signs.

$$\varphi = (\mu - 1)(\varrho - \varrho').$$

In the *meniscus* both are negative, and therefore

$$\varphi = -(\mu - 1)(\varrho - \varrho').$$

In the *double concave* lens ϱ is positive and ϱ' negative, and

$$\varphi = (\mu - 1)(\varrho + \varrho').$$

In the *double convex* ϱ is negative and ϱ' positive, and

$$\varphi = -(\mu - 1)(\varrho + \varrho').$$

In the *plano-concave* ϱ is positive and $\varrho' = 0$; wherefore

$$\varphi = (\mu - 1)\varrho.$$

In the *plano convex* ϱ is negative and $\varrho' = 0$, and

$$\varphi = -(\mu - 1)\varrho.$$

From this it appears that, when the medium of which the lens is composed is denser than the surrounding medium, or $\mu > 1$, the value of φ is positive in the double concave, the plano-concave, and the concavo-convex lenses; and therefore these lenses cause parallel rays to *diverge:* while, in the double-convex, the plano-convex, and the meniscus, φ being negative, parallel rays are made to *converge.* When the medium of which the lens is composed is *rarer* than the surrounding medium, μ is less than unity, and the sign of φ is reversed. Accordingly, convex lenses of a rarer medium give divergence to parallel rays, and concave lenses convergence.

In the double concave or double convex of equal radii, $\varrho' = \varrho$, and

$$\varphi = \pm(\mu - 1)2\varrho;$$

or the power of such a lens is double that of the plano-concave or plano-convex of the same curvature.

When the lens is of glass $\mu = \frac{3}{2}$, and therefore $\mu - 1 = \frac{1}{2}$, nearly; wherefore

$$\varphi = \tfrac{1}{2}(\varrho - \varrho').$$

In the case of the double concave or double convex lenses, in which ϱ and ϱ' are of opposite signs, $\varphi = \pm\frac{1}{2}(\varrho + \varrho')$; *i. e.* the power of such a lens is an arithmetical mean between its curvatures, or its focal length an harmonic mean between its radii. When these radii are equal $\varphi = \varrho$, or the focal length is equal to radius.

(158). If we differentiate the equation,

$$\alpha' = \alpha + \varphi,$$

we have $d\alpha' = d\alpha$; from which there is $d\delta' = \frac{\delta'^2}{\delta^2} . d\delta$. Whence it follows that the two foci move always in the same direction, $d\delta'$ and $d\delta$ being of the same sign; and the rates of their progress are as the squares of their distances from the lens.

It is easy to trace the corresponding positions of these foci. First, let the lens be of the concave kind, in which φ is always positive; then, as long as α is positive, α' is also positive, and greater than it; and while α decreases from infinity to nothing, α' decreases from infinity to φ. When α, passing through nothing, becomes negative, $\alpha' = \varphi - \alpha$ still continues positive, until $\alpha = \varphi$, when it becomes nothing.

Finally, when $\alpha > \varphi$, α' becomes negative; and as α increases negatively, α' increases also without limit.

From all this we learn, that when the incident rays diverge, the refracted rays diverge still more; when the former become parallel, the latter diverge from the principal focus; and when the divergence of the incident rays is changed into convergence, the divergence of the refracted rays still continues, though less than before, until, when the incident rays converge to the principal focus of rays proceeding in an opposite direction, the refracted rays become parallel; and, finally, their divergence is changed into convergence, when the point to which the incident rays converge is still nearer to the lens.

If the lens be of the convex kind, φ is negative, and the equation is

$$\alpha' = \alpha - \varphi;$$

an equation differing from the former in the signs of α and α'. Wherefore all that has been said of concave lenses may be applied directly to convex, if we only substitute convergence for divergence, and *v. v.* Hence, as the lenses of the concave kind increase the divergence and diminish the convergence, so those of a convex character increase the convergence and diminish the divergence.

(159.) From the equation $\alpha' = \alpha + \varphi$, several useful relations between the distances of the conjugate foci may be derived.

For, if δ and δ' denote these distances, and f the focal length of the lens, that equation is equivalent to

$$\frac{1}{\delta'} = \frac{1}{\delta} + \frac{1}{f}, \quad \text{whence } \frac{\delta}{\delta'} = \frac{\delta + f}{f};$$

i. e. the distance of the radiant from the lens is to the distance of its conjugate from the same, as the distance of the radiant from the principal focus of rays coming in an opposite direction is to the focal length.

Again, from these equations we deduce

$$\delta - \delta' = \frac{\delta^2}{\delta + f};$$

i. e. the distance of the radiant from the lens is a mean proportional between the distance of the radiant from its conjugate and from the principal focus of rays proceeding in an opposite direction.

Hence, when the position of the radiant and its conjugate are given, we may find the position of the lens. For in this case $\delta - \delta'$, the distance between the foci, is given; and δ, the distance of the radiant from the lens, is required; wherefore, denoting the former by a, and the latter by x, from the preceding result we obtain the equation

$$x^2 - ax - af = 0, \quad \text{whence } x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + af};$$

which determines the two positions of the lens which satisfy the problem. When a and f have the same sign, the problem is always possible. When they are of different signs, it will become impossible when $a < 4f$.

From this we learn that the least distance between the foci is equal to four times the focal length; a result which might readily have been obtained directly.

(160.) In what has preceded, we have supposed the thickness of the lens so small that it may be neglected in our calculations, a supposition which has greatly simplified our results. To complete the theory of refraction by a single lens, however, it will be necessary to consider the effect of the thickness, when that thickness is too considerable to be neglected.

The vergencies of the pencil, before and after refraction by the 1st and 2d surface respectively, being denoted by α and β, β' and α'; we have, as before, the equations

$$\mu(\beta - \rho) = \alpha - \rho,$$
$$\mu(\beta' - \rho') = \alpha' - \rho'.$$

Now, the distances of the focus of the pencil, after the first refraction, from the 1st and 2d surfaces, respectively, differ from one another by the interval between those surfaces or the thickness of the lens; wherefore, if θ denote the reciprocal of that thickness, there is

$$\frac{1}{\beta'} - \frac{1}{\beta} = \frac{1}{\theta},$$

and it remains to eliminate β and β'.

To effect this elimination, the equation last found may be put under the form

$$\beta'\beta + (\beta' - \beta)\theta = 0;$$

and if we multiply this equation by μ^2, and substitute for $\mu\beta$, $\mu\beta'$, their values derived from first two equations, we find

$$\left[\alpha + (\mu - 1)\rho\right]\left[\alpha' + (\mu - 1)\rho'\right] + \mu\theta\left[(\alpha' - \alpha) + (\mu - 1)(\rho' - \rho)\right] = 0;$$

an equation which expresses the relation between α and α', whatever be the thickness of the lens.

If we divide this equation by θ, and make $\frac{1}{\theta} = 0$ in the result, we have, as before, for a lens of inconsiderable thickness,

$$\alpha' - \alpha + (\mu - 1)(\rho' - \rho) = 0.$$

(161.) When the incident rays are parallel, $\alpha = 0$; and if the resulting value of α' be denoted by φ, there is

$$\varphi = (\mu - 1)\frac{\mu\theta(\rho - \rho') - (\mu - 1)\rho\rho'}{\mu\theta + (\mu - 1)\rho};$$

the general expression of the power of a lens of any thickness.

When the lens, whose curvatures are ρ and ρ', is turned in the opposite direction with respect to the incident light, it is evident that ρ' becomes ρ, and *v. v.*, and that the signs of both are changed. It is obvious, from the inspection of the formula, that the value of φ does not remain the same after this substitution, unless in the case in which the two curvatures are equal and opposite; and, accordingly, in all lenses whose thickness is considerable, the double convex and double concave of equal curvatures excepted, the power of the lens is altered by reversing its position with respect to the incident light.

If we divide the numerator and denominator of this expression by $\mu\theta$, and denote the reciprocal of θ, or the thickness of the lens, by δ, it may be put under the form,

$$\varphi = (\mu - 1)\left[\frac{\rho}{1 + \frac{\mu - 1}{\mu}\rho\delta} - \rho'\right].$$

To obtain an approximate value of φ, when the thickness of the lens is small, we have only to develop the first term of the quantity within the brackets, either by the binomial theorem or actual division; and if we neglect all the powers of $\rho\delta$ except the first, we find

$$\varphi = (\mu - 1)\left[(\rho - \rho') - \frac{\mu - 1}{\mu}\rho^2\delta\right].$$

(162.) Returning to the general equation, (160.), if $\mu^2\theta^2$ be added and subtracted, the result will remain unchanged, and it will assume the following very symmetrical form:

$$\left[\alpha + (\mu - 1)\rho + \mu\theta\right]\left[\alpha' + (\mu - 1)\rho' - \mu\theta\right] + \mu^2\theta^2 = 0.$$

As an application of this formula let us take the case of the *sphere*. Here the curvatures of the two surfaces, ρ and ρ', are equal, the former being negative and the latter positive; and $\theta = \frac{1}{2}\rho$. Wherefore the equation becomes

$$\left[\alpha - \left(\frac{\mu}{2} - 1\right)\rho\right]\left[\alpha' + \left(\frac{\mu}{2} - 1\right)\rho\right] + \frac{\mu^2}{4}\rho^2 = 0.$$

When the incident rays are parallel, $\alpha = 0$; and, denoting the resulting value of α' by φ, we find

$$\varphi = \frac{\mu - 1}{\mu - 2} \cdot 2\rho.$$

(163.) In a *plano-spherical* lens, whose *plane* surface is turned towards the incident light, $\rho = 0$, and omitting the trait in the symbol of the curvature of the 2d surface, the equation of the preceding article is reduced to

$$(\alpha + \mu\theta)\left[\alpha' + (\mu - 1)\rho\right] - \mu\theta\alpha = 0.$$

When the incident rays are parallel, $\alpha = 0$, and we have

$$\varphi = -(\mu - 1)\rho, \quad \text{or } f = \frac{-r}{\mu - 1}.$$

Wherefore the focal length of a lens of this kind is altogether independent of the thickness; a result which is otherwise evident, inasmuch as the rays undergo no refraction at the 1st surface.

In a *plano-spherical* lens, whose *curved surface* is turned towards the incident light, $\rho' = 0$, and the equation becomes

$$\left[\alpha + (\mu - 1)\rho\right](\alpha' - \mu\theta) + \mu\theta\alpha' = 0.$$

For parallel rays, $\alpha = 0$; and there is

$$\varphi\left[\mu\theta + (\mu - 1)\rho\right] = \mu\theta(\mu - 1)\rho;$$

or, substituting for φ, θ, and ρ, their reciprocals, f, δ, and r;

$$f = \frac{\delta}{\mu} + \frac{r}{\mu - 1}.$$

It is evident that, if r denote the radius of the spherical surface in a lens of this kind having its curved surface turned towards the incident light, it becomes $-r$ when the lens is turned in the opposite direction. Accordingly, if f and f' denote the focal lengths of the lens in the two cases,

$$f = \frac{\delta}{\mu} + \frac{r}{\mu - 1}, \quad \text{and } f' = \frac{r}{\mu - 1};$$

and subtracting,

$$f - f' = \frac{\delta}{\mu}.$$

That is, the difference between the focal lengths of the lens in the two cases is to its thickness in the constant ratio of the sine of refraction to the sine of incidence. If the lens be of crown glass, the difference of the two focal lengths is two-thirds of the thickness.

When the lens is a *hemisphere*, having its curved surface turned towards the incident light, r is negative, and $\delta = r$; wherefore, substituting in the equation last obtained, we find

$$f = \frac{-r}{\mu(\mu - 1)};$$

and, accordingly, the focal length of the hemisphere, in this case, is to $\frac{-r}{\mu - 1}$, the value of the focal length when the plane side is turned towards the incident light in the ratio of 1 to μ.

(164.) When the two surfaces of the lens are *concentric*, the problem is a case of that already investigated, in which, namely, the thickness of the lens is equal to the sum or difference of the radii of the two surfaces, according as the directions of their curvatures are opposed or coincident. The investigation of this case, however, will be more simple, if it be derived immediately from equation (145.) in which the distances are referred to the centre.

Let u and v denote the reciprocals of the distances of the foci from the centre, after refraction by the first surface, v' and u', the analogous quantities for the second refraction, and ρ and ρ', the curvatures of the surfaces; then we have

$$v - \rho = \mu(u - \rho), \qquad v' - \rho' = \mu(u' - \rho').$$

Now, subtracting the former equation from the latter, and observing that $v = v'$, there is $\rho - \rho' = \mu(u' - u + \rho - \rho')$: whence

$$u' - u = \frac{\mu - 1}{\mu}(\rho' - \rho),$$

a result which is true, whatever be the interval between the surfaces.

When the curvatures of the two surfaces are turned the *same way*, and consequently the surfaces themselves *parallel*, ρ and ρ' are together positive, or together negative, according as the lens is concave or convex towards the incident light. The reader cannot fail to observe a remarkable analogy between this formula, for the refraction of a pencil of rays by a medium bounded by parallel spherical surfaces, and that of (119.), in which the bounding surfaces are parallel planes.

When the curvatures of the two surfaces lie in *opposite* directions, the solid which they contain must necessarily be of the form of a *double convex lens*; and, therefore, the sign of ϱ, the curvature of the first surface, becomes negative. In this case, therefore, the formula is

$$u' - u = \frac{\mu - 1}{\mu}(\varrho + \varrho').$$

(165.) In the case of a *sphere*, $\varrho' = \varrho$, and the equation of the preceding article becomes

$$u' - u = \frac{\mu - 1}{\mu} 2\varrho.$$

Hence, if φ denote the reciprocal of the principal focal length of the sphere, estimated from the centre, or the value of u' when $u = 0$,

$$\varphi = \frac{\mu - 1}{\mu} 2\varrho;$$

and, substituting this in the preceding equation, it becomes

$$u' - u = \, .$$

If the sphere be of *glass*, $\mu = \frac{3}{2}$, nearly; wherefore

$$\varphi = \tfrac{2}{3}\varrho, \qquad \text{or } f = \tfrac{3}{2} r.$$

If the sphere be of *water*, $\mu = \frac{4}{3}$, nearly; wherefore

$$\varphi = \tfrac{1}{2}\varrho, \quad \text{and } \therefore f = 2r.$$

Hence the principal focus of a sphere of glass is distant from it by half the radius; that of a sphere of water by the entire radius.

(166.) We may now proceed to consider the refraction of a small pencil of rays by any combination of lenses, whose axes are coincident. We shall limit our attention to lenses of inconsiderable thickness; and take, in the first instance, the case in which these lenses are placed in contact.

Let α denote the *vergency* of the pencil incident upon the system; α', α'', α''', &c. $\alpha^{(n)}$, the *vergencies* of the refracted pencil, after refraction by the 1st, 2d, 3d, &c. nth lenses, severally; and φ', φ'', φ''', &c. $\varphi^{(n)}$, the *powers* of these lenses. Then, since the lenses are in contact, the focal distance of the pencil after refraction by any lens of the system becomes that of the pencil incident upon the next; and we have, therefore, the following equations:

$$\begin{aligned} \alpha' - \alpha &= \varphi' \\ \alpha'' - \alpha' &= \varphi'' \\ \alpha''' - \alpha'' &= \varphi''' \\ \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \\ \alpha^{(n)} - \alpha^{(n-1)} &= \varphi^{(n)}; \end{aligned}$$

and adding,

$$\alpha^{(n)} - \alpha = \varphi' + \varphi'' + \varphi''' + \&c. + \varphi^{(n)};$$

an equation which expresses the relation between the *vergencies* of the incident and refracted pencils, the *powers* of the several lenses which compose the system being known. The powers of the component lenses are expressed in terms of the curvatures of their surfaces, and the index of refraction of the substance of which the lens is composed, by means of the formula obtained (156).

(167.) If φ denote the *power* of the system, or the value of $\alpha^{(n)}$ when $\alpha = 0$,

$$\varphi = \varphi' + \varphi'' + \varphi''' + \&c. + \varphi^{(n)}.$$

And substituting this in the preceding equation, it becomes

$$\alpha^{(n)} - \alpha = \varphi.$$

From the former of these equations we learn that "the power of any system of lenses placed in contact is the sum of the powers of the component lenses." The word *sum* denoting, as in common algebra, that the quantities are to be affected with their proper signs.

With respect to the signs of these quantities, it is obvious from (157.) that the powers of all lenses of the *concave* kind are *positive*, those of lenses of the *convex* kind, *negative*.

The latter of these equations is precisely analogous to the formula for a single lens, and informs us that "the difference between the vergencies of the incident and refracted pencils is a constant quantity, being equal to the vergency given to parallel rays by refraction through the system." The *positive* sign, affecting these quantities, denotes *divergence*, and the *negative*, *convergence*.

(168.) When the lenses are separated by any intervals, the vergency of the refracted pencil, after refraction by any lens of the system, is no longer that of the pencil incident upon the following:—Let α and β denote the vergency of the pencil before and after refraction by the 1st lens; α' and β', the analogous quantities for the 2d; α'' and β'', for the 3d, &c. Also, let φ, φ', φ'', &c. denote the powers of the several lenses, and θ, θ', θ'', &c. the reciprocals of the intervals between them; then, reasoning as in (160.), we obtain the following relations:

$$\beta - \alpha = \varphi, \quad \beta' - \alpha' = \varphi', \quad \beta'' - \alpha'' = \varphi'', \quad \text{\&c.}$$

$$\frac{1}{\alpha'} - \frac{1}{\beta} = \frac{1}{\theta}, \quad \frac{1}{\alpha''} - \frac{1}{\beta'} = \frac{1}{\theta'}, \quad \text{\&c.}$$

And if the quantities β, α', β', α'', &c. be eliminated by means of these equations, the resulting equation will express the relation between the vergency of the pencil before and after refraction by the system.

Thus, in the case of two lenses, we have the three equations:

$$\beta - \alpha = \varphi, \quad \beta' - \alpha' = \varphi', \quad \frac{1}{\alpha'} - \frac{1}{\beta} = \frac{1}{\theta}.$$

It is easily seen that the last of these equations may be put under the form

$$(\theta - \alpha')(\theta + \beta) = \theta^2;$$

and substituting in this, for β and α', their values, $\alpha + \varphi$, and $\beta' - \varphi'$, obtained from the two former equations, and omitting the trait over β', as no longer necessary, we obtain

$$(\theta + \varphi' - \beta)(\theta + \varphi + \alpha) = \theta^2.$$

When the incident rays are parallel, $\alpha = 0$; and denoting the resulting value of β, or the power of the lens, by Φ, we have $(\theta + \varphi)(\theta + \varphi' - \Phi) = \theta^2$. Whence, denoting the reciprocals of φ, φ', and Φ, by f, f', and F; and the reciprocal of θ, or the thickness of the lens, by δ, there is

$$\text{F} = \frac{(\delta + f)f'}{\delta + f + f'},$$

the focal length of the compound lens, expressed in terms of the focal lengths of the simple lenses and of the interval between them.

IV.

Of Aberration in Refraction by Lenses.

(169). If α and β denote the reciprocals of the distances of the intersections of the incident and refracted rays with the axis, referred to the first surface of the lens, β' and α' the analogous quantities for the 2d surface, and θ the reciprocal of the interval between these surfaces, or of the thickness of the lens, the relations which exist amongst the ultimate values of these quantities, when the aperture of the refracting surfaces is evanescent, are determined by the equations

$$\alpha - \rho = \mu(\beta - \rho), \quad \alpha' - \rho' = \mu(\beta' - \rho'), \quad \frac{1}{\beta'} - \frac{1}{\beta} = \frac{1}{\theta}.$$

From these equations we have obtained, by elimination, the relation between α and the *ultimate* value of α', when the aperture of the lens is *evanescent* (160). We now proceed to inquire the difference between the ultimate value of α' and its *approximate* value when the aperture is *finite*, though small.

Let x and x' denote the semi-apertures of the 1st and 2d surfaces, respectively; then it is evident that α', the quantity whose variation is sought, is a function of two variables, β' and x'. Wherefore we have

$$d\alpha' = \left(\frac{d\alpha'}{d\beta'}\right)d\beta' + \left(\frac{d\alpha'}{dx'}\right)dx',$$

the quantities within the brackets denoting the partial differential coefficients of α', with respect to the two quantities upon which it depends.

Now, if we differentiate the 2d and 3d of the preceding equations, we obtain

$$\left(\frac{d\alpha'}{d\beta'}\right) = \mu, \quad d\beta' = \frac{\beta'^2}{\beta^2}.d\beta.$$

But, if the variation of the quantities β and α', arising from the change of aperture only, be denoted by $\varkappa.x^2$, $\varkappa'.x'^2$, respectively, there is

$$d\beta = \varkappa x^2, \text{ whence } \left(\frac{d\alpha'}{d\beta'}\right)d\beta' = \mu\frac{\beta'^2}{\beta^2}\varkappa.x^2.$$

Also, $$\left(\frac{d\alpha'}{dx'}\right)dx' = \varkappa'.x'^2 = \varkappa'.\frac{\beta^2}{\beta'^2}x^2;$$

since $x' = \frac{\beta}{\beta'}.x$. And, substituting these values in the expression of $d\alpha'$, we find

$$d\alpha' = \left(\mu.\frac{\beta'^2}{\beta^2}.\varkappa + \frac{\beta^2}{\beta'^2}.\varkappa'\right)x^2.$$

In which it only remains to substitute for $\varkappa$ and $\varkappa'$ their values, given in the second section of this chapter.

To proceed with these substitutions, if we substitute β for α', and μ for m, in the value of $d\alpha'$ (149.), it becomes

$$d\beta = \varkappa.x^2 = \tfrac{1}{2}(\beta - \varrho)^2(\beta - \mu\alpha)x^2.$$

Whence, eliminating ϱ by means of the first of the equations written in the commencement of this article, there is

$$\varkappa = \frac{1}{2(\mu-1)^2}(\beta - \alpha)^2(\beta-\mu\alpha).$$

The value of $\varkappa'$ is obtained from this by substituting β' and α' for α and β, respectively, and $\frac{1}{\mu}$ for μ; and therefore

$$\varkappa' = \frac{-\mu}{2(\mu-1)^2}(\beta' - \alpha')^2(\beta' - \mu\alpha').$$

And substituting in the expression of $d\alpha'$, given above,

$$d\alpha' = \frac{\mu x^2}{2(\mu-1)^2}\left\{\frac{\beta'^2}{\beta^2}(\beta-\alpha)^2(\beta-\mu\alpha) - \frac{\beta^2}{\beta'^2}(\beta'-\alpha')^2(\beta'-\mu\alpha')\right\},$$

the general expression for $d\alpha'$, whatever be the thickness of the lens.

The relations among the quantities, α, α', β, β', which enter this expression, are determined by the three equations written above. If β and β' be eliminated by means of the first two of these equations, the resulting value of $d\alpha'$ will be expressed as a symmetrical function of α, ϱ, and α', ϱ'.

(170.) When the thickness of the lens may be neglected as inconsiderable, the third of these equations becomes

$$\beta' = \beta.$$

And substituting in the other two, and in the expression for $d\alpha'$, obtained in the preceding article, there is

$$\alpha - \varrho = \mu(\beta - \varrho), \qquad \alpha' - \varrho' = \mu(\beta - \varrho'),$$

$$d\alpha' = \frac{\mu x^2}{2(\mu - 1)^2} \left\{ (\beta - \alpha)^2(\beta - \mu\alpha) - (\beta - \alpha')^2(\beta - \mu\alpha') \right\},$$

equations which contain every thing requisite to the development of the theory of the aberration of a thin lens.

If, in the expression for $d\alpha'$, just obtained, the quantity within the brackets be developed by performing the multiplications indicated, it becomes

$$(\mu + 2)(\alpha' - \alpha)\beta^2 - (2\mu + 1)(\alpha'^2 - \alpha^2)\beta + \mu(\alpha'^3 - \alpha^3)$$
$$= (\alpha' - \alpha)\left[(\mu + 2)\beta^2 - (2\mu + 1)(\alpha + \alpha')\beta + \mu(\alpha^2 + \alpha\alpha' + \alpha'^2)\right];$$
$$\therefore\ d\alpha' =$$
$$\frac{\mu(\alpha' - \alpha)}{2(\mu - 1)^2} x^2 \left\{ (\mu + 2)\beta^2 - (2\mu + 1)(\alpha + \alpha')\beta + \mu(\alpha^2 + \alpha\alpha' + \alpha'^2) \right\},$$

a result which will be found convenient in its application hereafter.

(171.) To express $d\alpha'$ as a function of the vergencies of the incident and refracted pencils and the curvatures of the two surfaces, we have only to eliminate β by means of the first two equations of the preceding article. Thus, from the former we obtain

$$\beta - \alpha = \frac{\mu - 1}{\mu}(\varrho - \alpha), \qquad \beta - \mu\alpha = \frac{\mu - 1}{\mu}\left[\varrho - (\mu + 1)\alpha\right],$$

and, as it is evident from the second of these equations, that $\beta - \alpha'$, $\beta - \mu\alpha'$, are similar functions of ϱ' and α', substituting in the former expression of $d\alpha'$, obtained above, there is

$$d\alpha' = \frac{\mu - 1}{2\mu^2} x^2 \left\{ (\varrho - \alpha)^2[\varrho - (\mu + 1)\alpha] - (\varrho' - \alpha')^2[\varrho' - (\mu + 1)\alpha'] \right\};$$

an expression remarkable for its symmetry, and which, combined with the equation,

$$\alpha' - \alpha = (\mu - 1)(\varrho - \varrho'),$$

obtained by eliminating β from the first two equations of the preceding article, contains the whole theory of a thin lens.

(172.) When the incident rays are *parallel*,

$$\alpha = 0, \text{ and } \alpha' = \varphi = (\mu - 1)(\varrho - \varrho');$$

and these substitutions being made in the preceding expression for $d\alpha'$, we find

$$d\phi = \frac{\mu - 1}{2\mu^2} x^2 \left\{ \varrho^3 + [\mu(\varrho - \varrho') - \varrho]^2 [\mu^2(\varrho - \varrho') - \varrho] \right\}.$$

If the quantity within the brackets in this expression be developed, it becomes

$$\mu(\varrho - \varrho') \left\{ \mu^3(\varrho - \varrho')^2 - \mu(2\mu + 1)(\varrho - \varrho')\varrho + (\mu + 2)\varrho^2 \right\}$$
$$= \mu(\varrho - \varrho') \left\{ (2 - 2\mu^2 + \mu^3)\varrho^2 + (\mu + 2\mu^2 - 2\mu^3)\varrho\varrho' + \mu^3\varrho'^2 \right\}:$$
$$\therefore d\phi =$$
$$\frac{\mu - 1}{2\mu}(\varrho - \varrho')x^2 \left\{ (2 - 2\mu^2 + \mu^3)\varrho^2 + (\mu + 2\mu^2 - 2\mu^3)\varrho\varrho' + \mu^3\varrho'^2 \right\}.$$

In a *plano-spherical* lens, having its plane side turned towards the incident light, $\varrho = 0$, and omitting the *trait* in the symbol of the curvature of the second surface, the expression of $d\varphi$ becomes

$$d\phi = -\frac{\mu^2(\mu - 1)}{2} \varrho^3 x^2.$$

To express this in terms of the power of the lens, we have only to eliminate ϱ by means of the relation $\phi = -(\mu - 1)\varrho$, and there is

$$d\phi = \tfrac{1}{2}\left(\frac{\mu}{\mu - 1}\right)^2 \phi^3 x^2.$$

In a *plano-spherical* lens, having its curved side turned towards the incident light, $\varrho' = 0$, and there is

$$d\phi = \frac{\mu - 1}{2\mu}(\mu^3 - 2\mu^2 + 2)\varrho^3 x^2.$$

Or, eliminating ϱ by means of the equation $\phi = (\mu - 1)\varrho$,

$$d\phi = \frac{\mu^3 - 2\mu^2 + 2}{2\mu(\mu - 1)^2}\phi^3 x^2.$$

To compare the values of $d\alpha'$ in the same plano-spherical lens turned in opposite directions with respect to the incident light, we have only to consider the curvatures as equal and affected with opposite signs in the preceding expressions, and it will be readily seen that the value of $d\phi$, when the plane surface is turned towards the incident light, is greater than in the contrary position in the ratio of $\mu^3 : \mu^3 - 2\mu^2 + 2$.

In a double convex, or double concave lens of equal curvatures, $\varrho' = -\varrho$, and the value of $d\phi$ becomes

$$d\phi = \frac{\mu - 1}{\mu}\left(4\mu^3 - 4\mu^2 - \mu + 2\right)\varrho^3 x^2.$$

Or, since in this case $\phi = (\mu - 1)2\varrho$,

$$d\phi = \frac{4\mu^3 - 4\mu^2 - \mu + 2}{8\mu(\mu - 1)^2}\,\phi^3 x^2.$$

When the lens is of crown glass, in which $\mu = \frac{3}{2}$ nearly, the coefficients of $\phi^3 x^2$, in the preceding values of $d\phi$, are respectively $\frac{9}{2}$, $\frac{7}{6}$, and $\frac{5}{3}$. If the three lenses, then, have the same apertures and powers, the quantity $d\phi$ is greatest in the plano-spherical lens having its plane surface turned towards the incident light, and least in the same lens turned in the opposite way.

(173.) When the *emergent* rays are parallel,

$$\alpha' = 0, \text{ and } \alpha = -(\mu - 1)(\varrho - \varrho');$$

and these values being substituted in the equation of (171.), we find

$$d\alpha' = \frac{\mu - 1}{2\mu^2}\,x^2\left\{[\mu(\varrho - \varrho') + \varrho']^2[\mu^2(\varrho - \varrho') + \varrho'] - \varrho'^3\right\},$$

a result which agrees with that obtained in the preceding article, if we substitute ϱ' for ϱ, and *v. v.*, and change the signs of both quantities; that is, in fact, if the lens be turned in the opposite direction with respect to the incident light. Hence it appears that, when the refracted rays are parallel, the value of $d\alpha'$ is the same as in the case in which the incident rays are parallel, if the lens be turned in the opposite direction with respect to the incident light in the two cases. This observation will be found of importance hereafter, when we come to speak of the simple microscope.

(174.) We shall now return to the general expression of $d\alpha'$ for a lens of inconsiderable thickness (170.).

The distances of the radiant and its conjugate being given, let it be required to determine the form of the lens for which the value of $d\alpha'$ is a *minimum*.

Here α and α' are given, and β is the variable which is to be determined by the conditions of the question. Therefore differentiating the quantity within the brackets in the expression of $d\alpha'$ (170.) with respect to β, and equating the result to nothing, we obtain

$$\beta = \frac{2\mu+1}{2(\mu+2)}(\alpha + \alpha').$$

And this value of β being substituted in the expression of $d\alpha'$ itself, the quantity within the brackets becomes

$$\frac{1}{\mu+2}\left\{(\mu - \tfrac{1}{4})(\alpha^2 + \alpha'^2) - (\mu^2 + \tfrac{1}{2})\alpha\alpha'\right\}$$

$$= \frac{1}{\mu+2}\left\{(\mu - \tfrac{1}{4})(\alpha' - \alpha)^2 - (\mu - 1)^2\alpha\alpha'\right\};$$

wherefore the minimum value of $d\alpha'$ is

$$d\alpha' = \frac{\mu(\alpha'-\alpha)}{2(\mu+2)}x^2\left\{\frac{\mu-\frac{1}{4}}{(\mu-1)^2}(\alpha' - \alpha)^2 - \alpha\alpha'\right\}.$$

To find the form of the lens, we have only to substitute the value of β, obtained above, in the equations,

$$(\mu - 1)\varrho = \mu\beta - \alpha, \qquad (\mu - 1)\varrho' = \mu\beta - \alpha',$$

which are equivalent to the first two equations (170.); and making for abbreviation,

$$p = \frac{2\mu^2+\mu}{2(\mu-1)(\mu+2)}, \qquad q = \frac{2\mu^2-\mu-4}{2(\mu-1)(\mu+2)},$$

we obtain

$$\varrho = p\alpha' + q\alpha$$

$$\varrho' = p\alpha + q\alpha'.$$

(175.) When the incident rays are parallel, $\alpha = 0$, and $\alpha' = \varphi$; and the minimum value of $d\alpha'$ becomes

$$d\varphi = \frac{\mu(\mu - \frac{1}{4})}{2(\mu+2)(\mu-1)^2}\varphi^3 x^2.$$

And for a lens of crown glass, $\mu = \frac{3}{2}$ nearly, and

$$d\phi = \tfrac{15}{14}\varphi^3 x^2.$$

With respect to the form of the lens in this case, it is determined by the equations,

$$\varrho = p\phi, \quad \varrho' = q\phi.$$

And, accordingly, the ratio of the curvatures of the two surfaces is independent of the power of the lens, and is

$$\frac{\varrho'}{\varrho} = \frac{q}{p} = \frac{2\mu^2 - \mu - 4}{2\mu^2 + \mu}.$$

When $\mu = \frac{3}{2}$, this ratio becomes

$$\frac{\varrho'}{\varrho} = -\tfrac{1}{6}.$$

The curvatures of the two surfaces, therefore, in a lens of this kind formed of crown glass, lie in opposite directions; that is, the lens is either a double convex or double concave; and the curvature of the posterior surface is the one-sixth part of that of the anterior. Such a lens is called by artists a *crossed lens.*

If the index of refraction be of such a value as to satisfy the equation $2\mu^2 - \mu - 4 = 0$, that is, if

$$\mu = \frac{1 + \sqrt{33}}{4} = 1.686, \text{ nearly,}$$

which is about the value of μ for the more refrangible kinds of glass; then $\varrho' = 0$, and the form of the lens, for which the value of $d\phi$ is least, will be a plano-spherical lens having its curved surface turned towards the incident light.

The power of this lens is

$$\phi = (\mu - 1)\varrho = .686,\ \varrho.$$

It is evident from the observation made in (173.), that the best form of the lens, when the emergent rays are parallel, is the same as that which we have been just investigating, but reversed in position with respect to the incident light. This will appear also directly if we make $\alpha' = 0$, $\alpha = -\varphi$, in the equations of the preceding article; for thus the value of $d\alpha'$ is the same as before, and for the curvatures of the surfaces there is

$$\rho = -q\phi, \quad \rho' = -p\phi.$$

(176.) The form of the lens, which has been investigated in the preceding articles, is obviously the best form for a single lens of indefinitely small thickness, inasmuch as the aberration is less than in any other lens having the same radiant and conjugate. We shall, therefore, in what follows, take this as the *standard lens*, and compare the aberration and form of any other lens with its aberration and form.

It has been already found that the value of β, in the lens of best form, is $\frac{2\mu+1}{2(\mu+2)}(\alpha + \alpha')$; therefore in any other lens let us take

$$\beta = \frac{2\mu+1}{2(\mu+2)}(\alpha + \alpha') + \delta;$$

and substituting in the value of $d\alpha'$ (170.), the quantity within the brackets becomes

$$\frac{1}{\mu+2}\Big[(\mu - \tfrac{1}{4})(\alpha' - \alpha)^2 - (\mu - 1)^2\alpha\alpha' + (\mu + 2)^2\delta^2\Big]$$

$$= \frac{(\mu-1)^2}{\mu+2}\Big[\frac{\mu-\frac{1}{4}}{(\mu-1)^2}(\alpha' - \alpha)^2 - \alpha\alpha' + \varepsilon^2\Big],$$

making $(\mu + 2)\delta = (\mu - 1)\varepsilon$. Wherefore making, for the sake of abbreviation,

$$\frac{\mu}{\mu+2} = m, \quad \frac{\mu-\frac{1}{4}}{(\mu-1)^2} = n,$$

the general expression of $d\alpha'$ is

$$d\alpha' = \tfrac{1}{2}m(\alpha' - \alpha)x^2\big[n(\alpha' - \alpha)^2 - \alpha\alpha' + \varepsilon^2\big];$$

an expression differing from that obtained (174.) merely in the additional term ε^2.

When the incident rays are parallel, $\alpha = 0$, and $\alpha' = \phi$, and the value of $d\alpha'$ becomes

$$d\phi = \tfrac{1}{2}m\phi x^2(n\phi^2 + \varepsilon^2).$$

When ε is very small, the general value of $d\alpha'$, here given, will be *greater* or *less* than the particular value just alluded to, and which is obtained from the preceding by making $\varepsilon = 0$, according as the quantity, $n(\alpha' - \alpha)^2 - \alpha\alpha'$, is *positive* or *negative;* and, accordingly, that value of $d\alpha'$ is a *minimum* in the former case, and in the latter a *maximum.* When the medium of which the lens is composed is *denser* than the surrounding medium, the least value of μ is unity; when *rarer*, its least known value is $\frac{1}{3}$; hence $n = \frac{\mu - \frac{1}{4}}{(\mu - 1)^2}$ is always positive, and therefore the quantity, $n(\alpha' - \alpha)^2 - \alpha\alpha'$, will be always *positive* when α and α' are of different signs, or the conjugate foci at opposite sides of the lens; when they lie at the same side, it will be positive or negative, according as $n(\alpha' - \alpha)^2$ is greater or less than $\alpha\alpha'$.

The quantity α', or the vergency of the refracted pencil, is increased or diminished by the effect of aberration, according as α' and $d\alpha'$ are of the same or of opposite signs. When the incident rays are parallel, since m and n are always positive, it is obvious that $d\phi$ and ϕ are necessarily of the same sign, and, therefore, that ϕ is always increased by the increase of aperture, or the intersection of the extreme ray always nearer the lens than the principal focus.

(177.) To obtain the form of the lens, we must substitute the value of β, namely,

$$\beta = \frac{2\mu + 1}{2(\mu + 2)}(\alpha + \alpha') + \frac{\mu - 1}{\mu + 2}\varepsilon,$$

in the equations

$$(\mu - 1)\varrho = \mu\beta - \alpha, \quad (\mu - 1)\varrho' = \mu\beta - \alpha';$$

and, using the symbols p, q, and m, as before, we find

$$\varrho = p\alpha' + q\alpha + m\varepsilon$$

$$\varrho' = p\alpha + q\alpha' + m\varepsilon;$$

from which equations, the relation between the form of any lens and that of best form is immediately perceived.

When the incident rays are parallel, $\alpha = 0$, $\alpha' = \phi$, and these equations become

$$\varrho = p\phi + m\varepsilon, \quad \varrho' = q\phi + m\varepsilon.$$

(178.) As an application of the preceding equations, let it be required to find the aberration in a lens of a given form.

In this case ϱ and ϱ' are given, as also α and α'; and therefore ε is determined from either of the equations of the preceding article. Accordingly, substituting its value thus determined in the expression of $d\alpha'$ (176.), the resulting value will be the quantity sought.

Thus, if the lens be a double concave or double convex of equal curvatures, $\varrho' = -\varrho$; wherefore, substituting this value, and adding together the equations (177.), there is

$$(p + q)(\alpha + \alpha') + 2m\varepsilon = 0;$$

$$\text{but } p + q = 2\frac{\mu+1}{\mu+2}, \quad m = \frac{\mu}{\mu+2};$$

$$\therefore \varepsilon = -\frac{\mu+1}{\mu}(\alpha + \alpha').$$

And this being substituted in (176.), the resulting value of $d\alpha'$ is that required.

To obtain the curvature of the surfaces, we have only to eliminate ε from the equations of the preceding article by subtracting the latter from the former, and we find

$$2\varrho = (p - q)(\alpha' - \alpha) = \frac{\alpha' - \alpha}{\mu - 1},$$

as is otherwise evident.

(179.) Conversely, let it be required to find the form of the lens which produces a given aberration.

This is done by equating the general value of $d\alpha'$ to the assigned quantity, and solving the resulting equation for ε; the value of ε, thus obtained, being substituted in the values of ϱ and ϱ' (177.), the form of the lens is determined.

Since ε occurs in the second dimension only in the expression of $d\alpha'$, it is evident that the equation by which it is determined will give two values, which are equal with opposite signs. Hence it appears that there are, in general, two lenses which, for a given distance of the radiant and its conjugate, produce a given aberration; and that the curvatures of the surfaces of these lenses are nearly related, differing only in the sign of ε. If the values of ε, determined as above, be imaginary, there is no lens which fulfils the conditions of the question.

It readily appears from the expression of $d\alpha'$ (176.), that the value of that quantity will remain unaltered if we substitute α' for α, and *v. v.*, and change the signs of α, α', and ε, simultaneously. But, if these changes be made in the values of ρ and ρ' (177.), ρ is changed into ρ', and *v. v.*; and the signs of both are changed. Hence it appears that if the focus of the refracted pencil be made that of the incident pencil, and the rays proceeding from it in the opposite direction be refracted back to the former focus of incident rays, the value of $d\alpha'$ will remain unaltered. This is a generalization of the result obtained (173.).

(180.) The position of the radiant and its conjugate being given, it is required to determine the form of the lens whose aberration is nothing.

To solve this problem, we have only to equate to nothing the quantity within the brackets in the expression of $d\alpha'$; and solving the resulting equation with respect to ε, we obtain

$$\varepsilon = \pm\sqrt{\left(\alpha\alpha' - n(\alpha' - \alpha)^2\right)};$$

and this value being substituted in the expressions for ρ and ρ', the form of the lens will be determined. Such a lens is termed *aplanatic.*

The value of ε is real, and therefore the problem possible, only when α and α' have the same sign, and $\alpha\alpha' > n(\alpha' - \alpha)^2$. In all other cases the problem is impossible. As the value of ε has the double sign, it follows that, when its value is real, there are two lenses aplanatic for the given position of the radiant and its conjugate.

When the incident rays are parallel, $\alpha = 0$, and $\alpha' = \phi$, and the quantity under the radical sign is reduced to $-n\phi^2$; and, since n is always positive, the value of ε is always imaginary, and it is therefore never possible to destroy the aberration of a single lens for parallel rays.

(181.) We now proceed to consider any combination of lenses, disposed in any manner along the same axis.

To begin with the case of two lenses: let α and β denote the reciprocals of the distances of the radiant and its conjugate for the first lens, α' and β' the analogous quantities for the second; also, let ϕ and ϕ' denote the powers of the two lenses, and θ the reciprocal of the interval between them. The relations which exist among the *ultimate* values of the quantities α, β, α', β', which they have when the apertures of the lenses are evanescent, are given by the equations

$$\beta - \alpha = \phi, \quad \beta' - \alpha' = \phi', \quad \frac{1}{\alpha'} - \frac{1}{\beta} = \frac{1}{\theta}.$$

But if x and x' denote the semi-apertures of the two lenses, it is evident that the general value of β' is a function of two variables, α' and x', and therefore that

$$d\beta' = \left(\frac{d\beta'}{dx'}\right)dx' + \left(\frac{d\beta'}{d\alpha'}\right)d\alpha';$$

$\left(\frac{d\beta'}{dx'}\right)$ and $\left(\frac{d\beta'}{d\alpha'}\right)$ denoting the partial differential coefficients of β' with respect to x' and α'. Now, differentiating the second and third of the preceding equations, we find

$$\left(\frac{d\beta'}{d\alpha'}\right) = 1, \quad d\alpha' = \frac{\alpha'^2}{\beta^2}\, d\beta;$$

and if the variations of β and β', arising from the aperture only, be denoted by $\varkappa . x^2$ and $\varkappa' . x'^2$, there is

$$d\beta = \varkappa . x^2, \quad \left(\frac{d\beta'}{dx'}\right)dx' = \varkappa' . x'^2.$$

Wherefore, substituting, we have

$$d\beta' = \varkappa' . x'^2 + \frac{\alpha'^2}{\beta^2}\, \varkappa . x^2.$$

And, accordingly, the total variation of β', after refraction by the two lenses, is equal to the variation of β' arising from the aperture, together with the analogous variation in β, produced by the first lens, multiplied by the fraction $\left(\frac{\alpha'}{\beta}\right)^2$.

If we observe that $x' = x . \frac{\beta}{\alpha'}$, the value of $d\beta'$ may be written

$$d\beta' = \left[\left(\frac{\alpha'}{\beta}\right)^2 \varkappa + \left(\frac{\beta}{\alpha'}\right)^2 \varkappa'\right] x^2;$$

in which it remains only to substitute for $\varkappa$ and $\varkappa'$ their values as found in the preceding part of this section.

(182.) If there be a third lens, the reciprocals of whose conjugate distances are denoted by α'' and β'', and the reciprocal of the distance between it and the second by θ', the relations which exist amongst the ultimate values of the distances are determined by the equations

$$\beta - \alpha = \phi, \quad \beta' - \alpha' = \phi', \quad \beta'' - \alpha'' = \phi'',$$

$$\frac{1}{\alpha'} - \frac{1}{\beta} = \frac{1}{\theta}, \quad \frac{1}{\alpha''} - \frac{1}{\beta'} = \frac{1}{\theta'}.$$

And the variation of β'' is

$$d\beta'' = \left(\frac{d\beta''}{dx''}\right)dx'' + \left(\frac{d\beta''}{d\alpha''}\right)d\alpha''.$$

But from the preceding equations we deduce

$$\left(\frac{d\beta''}{d\alpha''}\right) = 1, \quad d\alpha'' = \frac{\alpha''^2}{\beta'^2}\,d\beta'.$$

Substituting, therefore, and putting $\varkappa''\,.\,x''^2$ for $\left(\frac{d\beta''}{dx''}\right)dx''$, and for $d\beta'$ its value found in the preceding article,

$$d\beta'' = \varkappa''\,.\,x''^2 + \left(\frac{\alpha''}{\beta'}\right)^2 \varkappa'\,.\,x'^2 + \left(\frac{\alpha''\alpha'}{\beta'\beta}\right)^2 \varkappa\,.\,x^2;$$

or, since $x' = x\,.\,\frac{\beta}{\alpha'}$, $\quad x'' = x'\,.\,\frac{\beta'}{\alpha''} = x\,.\,\frac{\beta\beta'}{\alpha'\alpha''}$,

$$d\beta'' = \left[\left(\frac{\alpha'\alpha''}{\beta\beta'}\right)^2 \varkappa + \left(\frac{\beta\alpha''}{\alpha'\beta'}\right)^2 \varkappa' + \left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 \varkappa''\right] x^2.$$

In like manner, if there be a fourth lens, the reciprocals of the distances of the radiant and conjugate from which are denoted by α''' and β''', and in which the variation of β''', arising from aperture only, is denoted by $\varkappa'''\,.\,x'''^2$, the total variation of β''' will be

$$d\beta''' = \left[\left(\frac{\alpha'\alpha''\alpha'''}{\beta\beta'\beta''}\right)^2 \varkappa + \left(\frac{\beta\alpha''\alpha'''}{\alpha'\beta'\beta''}\right)^2 \varkappa' + \left(\frac{\beta\beta'\alpha'''}{\alpha'\alpha''\beta''}\right)^2 \varkappa'' + \left(\frac{\beta\beta'\beta''}{\alpha'\alpha''\alpha'''}\right)^2 \varkappa'''\right] x^2$$

the law of which is evident, and may be easily extended to any number of lenses.

(183.) When the lenses composing the system are *in contact*,

$$\alpha' = \beta, \quad \alpha'' = \beta', \quad \alpha''' = \beta'', \text{ \&c.}$$

In this case, therefore, the coefficients of $\varkappa$, $\varkappa'$, $\varkappa''$, &c. in the expressions just found, are all equal to unity; and for any number of lenses denoted by n there is

$$d\beta^{(n)} = [\varkappa + \varkappa' + \varkappa'' + \&c. + \varkappa^{(n)}]\,x^2;$$

in which it only remains to substitute for $\varkappa$, $\varkappa'$, $\varkappa''$, &c. their values as found above.

(184.) To obtain the conditions of *aplanatism* in any system of lenses, we have only to make the coefficient of x^2 equal to nothing in the values of $d\beta'$, $d\beta''$, &c. which have been just obtained.

Thus, in order that a combination of two lenses should be aplanatic, we have the equation of condition (181.),

$$\left(\frac{\alpha''}{\beta}\right)^2 \varkappa + \left(\frac{\beta}{\alpha'}\right)^2 \varkappa' = 0.$$

In a combination of three lenses, the equation of condition is

$$\left(\frac{\alpha'\alpha''}{\beta\beta'}\right)^2 \varkappa + \left(\frac{\beta\alpha''}{\alpha'\beta'}\right)^2 \varkappa' + \left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 \varkappa'' = 0;$$

and, in like manner, for any number of lenses; and it remains only to substitute for $\varkappa$, $\varkappa'$, $\varkappa''$, &c. their values furnished by the equations of the preceding articles.

When any number of lenses are combined in contact, the equation of condition becomes simply

$$\varkappa + \varkappa' + \varkappa'' + \&c. + \varkappa^{(n)} = 0.$$

To render any combination of lenses aplanatic is obviously an indeterminate problem; for there is but one equation of condition to be fulfilled, while there are as many unknown quantities, ε, ε', ε'', &c. as there are lenses. Hence it is evident that we are at liberty to superadd $(n - 1)$ arbitrary conditions to the problem, n denoting the number of lenses; and the equations expressing these conditions, together with the equation of aplanatism, will suffice to determine the quantities ε, ε', ε'', &c.; and it then remains only to substitute the values of these quantities, thus determined, in the expressions of the curvatures of the surfaces of the lenses (177.).

The simplest condition, and which first suggests itself, is to take each of the quantities, ε, ε', ε'', &c. (one excepted) equal to nothing; the remaining one will be determined by the equation of aplanatism. In this case, the aberration of all the lenses, but one, is the least possible for the respective distances of their foci.

Again, we might assume as the arbitrary conditions, that all the lenses, but one, should be equally curved on both sides. By this condition, the value of ε is determined in each of the lenses in question; for, if α and α' denote the vergencies of the incident and refracted rays in any one of these lenses, and μ its index of refraction, we have seen (178.) that in this case

$$\varepsilon = -\frac{\mu+1}{\mu}(\alpha + \alpha');$$

and similarly for the rest. These values of ε, ε', &c. being therefore substituted in the equation of condition obtained above, that equation will determine the remaining one.

(185.) We shall now proceed to develop the preceding theory in its simplest and most important application, namely, to the case of two lenses placed in contact; and inquire the forms of the two lenses, so that the combination shall be *aplanatic* for any assigned position of the radiant.

The condition of aplanatism in this case is

$$\varkappa + \varkappa' = 0,$$

$\varkappa$ and $\varkappa'$ being the coefficients of the square of the aperture in the values of $d\alpha'$ (176.); wherefore

$$\varkappa = \tfrac{1}{2}m(\alpha' - \alpha)\left[n(\alpha' - \alpha)^2 - \alpha\alpha' + \varepsilon^2\right],$$

α and α' denoting the vergencies of the incident and refracted pencils for the first lens. To obtain the value of $\varkappa'$, we have only to substitute α' and α'' for α and α', and m', n', and ε', for m, n, and ε, m' and n' being the same functions of μ', the index of refraction of the second lens, that m and n are of μ. In this manner we have

$$\varkappa' = \tfrac{1}{2}m'(\alpha'' - \alpha')\left[n'(\alpha'' - \alpha')^2 - \alpha'\alpha'' + \varepsilon'^2\right].$$

If the powers of the two lenses be denoted by ϕ and ϕ', there is

$$\alpha' - \alpha = \phi, \qquad \alpha'' - \alpha' = \phi'.$$

And if these values be substituted in the expressions of $\varkappa$ and $\varkappa'$, and the resulting expressions substituted in the equation of condition, $\varkappa + \varkappa' = 0$, it becomes

$$m\phi[\varepsilon^2 - \alpha\alpha' + n\phi^2] + m'\phi'[\varepsilon'^2 - \alpha'\alpha'' + n'\phi'^2] = 0;$$

which being fulfilled by means of the arbitrary quantities ε and ε', and the values of these quantities, thus determined, being substituted in the equations

$$\rho = p\alpha' + q\alpha + m\varepsilon, \qquad \rho' = p\alpha + q\alpha' + m\varepsilon,$$
$$\rho'' = p'\alpha'' + q'\alpha' + m'\varepsilon', \qquad \rho''' = p'\alpha' + q'\alpha'' + m'\varepsilon',$$

(in which ρ'' and ρ''' denote the curvatures of the surfaces of the second lens, and p' and q' the same functions of μ', that p and q are of μ), the forms of the lenses will be completely determined.

(186.) Since there are two arbitrary quantities, ε and ε', and but one equation to determine them, it is evident that we are at liberty to introduce into the problem some arbitrary condition, the choice of which will be of considerable importance *. Clairaut proposed to make the curvatures of the adjacent surfaces of the two lenses equal, one being convex and the other concave, in order that they might admit of being cemented together, and thus the loss of light, occasioned by reflexion at their surfaces, avoided. This condition is equivalent to $\rho'' = \rho'$, or substituting for ρ' and ρ'' their values,

$$p\alpha + q\alpha' + m\varepsilon = p'\alpha'' + q'\alpha' + m'\varepsilon',$$

by means of which one of the quantities, ε and ε', is determined in terms of the other. To this adaptation, however, there are weighty objections in practice; the compound lens will be liable to strain on the cooling of the cement, and moreover, a distortion of the same nature must ensue on every change

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* As the equation of condition is a quadratic with respect to ε and ε', it will evidently depend upon the assumed condition, whether these quantities admit of real values or not; that is, when the powers of the lenses are given, the condition introduced may render the problem impossible. When the powers of the lenses are not given, the problem will be possible under any condition whatever, since the resulting equation is of the third degree in the unknown quantity, ϕ or ϕ', and must therefore have at least one real root.

of temperature, if the two glasses be differently expansible by heat.

The most obvious condition seems to be to make one of the quantities ε and ε' equal to nothing, or to make one of the lenses of the best form (174.). This condition is equivalent to $d\kappa = 0$, or $d\kappa' = 0$; and in virtue of it the aberration of that lens will be a minimum for the particular distance of the radiant, and therefore any small change in that distance will not sensibly affect its value.

The foregoing seems naturally to suggest that if instead of making $d\kappa = 0$, or $d\kappa' = 0$, in one of the simple lenses, we were to take $d(\kappa + \kappa') = 0$, in the expression of the aberration of the compound, the combination would possess a considerable practical advantage. Accordingly, the coefficient $\kappa + \kappa'$ being differentiated with respect to α, our condition is

$$\frac{d\kappa}{d\alpha} + \frac{d\kappa'}{d\alpha} = 0.$$

Now, performing the operations indicated, and observing that $d\alpha'' = d\alpha' = d\alpha$, it becomes

$$m\phi\left[2\varepsilon.\frac{d\varepsilon}{d\alpha} - (\alpha + \alpha')\right] + m'\phi'\left[2\varepsilon'.\frac{d\varepsilon'}{d\alpha} - (\alpha' + \alpha'')\right] = 0.$$

But if we differentiate the values of ρ and ρ'' (177.) with respect to α, we find

$$m\frac{d\varepsilon}{d\alpha} + (p + q) = 0, \quad m'\frac{d\varepsilon'}{d\alpha} + (p' + q') = 0;$$

and substituting in the equation of condition for

$m\,\dfrac{d\varepsilon}{d\alpha}$, $m'\,\dfrac{d\varepsilon'}{d\alpha}$, their values $-(p + q)$, $-(p' + q')$, thus obtained, and making for abbreviation

$$2(p + q) = 4\frac{\mu+1}{\mu+2} = l, \quad 2(p' + q') = 4\frac{\mu'+1}{\mu'+2} = l',$$

it becomes

$$\varphi\big[l\varepsilon + m(\alpha + \alpha')\big] + \varphi'\big[l'\varepsilon' + m'(\alpha' + \alpha'')\big] = 0;$$

an equation which determines the relation between the arbitrary quantities ε and ε', when the aberration of the compound lens is a minimum.

Now, if this equation be fulfilled, together with that of the preceding article, by means of the arbitrary quantities ε and ε', and the values of these quantities, thus determined, substituted in the expressions of the curvatures of the surfaces (177.), the forms of the component lenses will be obtained. It is evident from what has been said, that the aberration of the compound lens will be nothing, and its differential also nothing; so that the combination possesses the advantage of being *aplanatic*, not only for the particular value of α, in virtue of the former of these equations, but also when that quantity receives any small variation, in virtue of the latter.

(187.) When the incident rays are parallel,

$$\alpha = 0, \qquad \alpha' = \varphi, \text{ and } \alpha'' = \varphi + \varphi',$$

and substituting, the equations of aplanatism are

$$m\varphi\big[\varepsilon^2 + n\varphi^2\big] + m'\varphi'\big[\varepsilon'^2 - \varphi(\varphi + \varphi') + n'\varphi'^2\big] = 0,$$

$$\varphi\big[l\varepsilon + m\varphi\big] + \varphi'\big[l'\varepsilon' + m'(2\varphi + \varphi')\big] = 0.$$

ε and ε' being determined by means of these equations, and the resulting values substituted in the equations

$$\varrho = p\varphi + m\varepsilon, \qquad \varrho'' = q'\varphi + p'(\varphi + \varphi') + m'\varepsilon',$$

$$\varrho' = q\varphi + m\varepsilon, \qquad \varrho''' = p'\varphi + q'(\varphi + \varphi') + m'\varepsilon',$$

the compound lens is completely determined, and will be aplanatic, not only for parallel rays, but also when the distance of the radiant is finite and considerable.

(188.) The quantity whose value we have been hitherto seeking is the variation of the reciprocal of the distance of the intersection of the emergent ray with the axis, arising from the aperture. If this quantity be denoted by $-\mathrm{K}x^2$, x being the semi-aperture of the first lens of the system, and the distance of the intersection of the emergent ray with the axis by δ, $d\left(\frac{1}{\delta}\right) = -\mathrm{K}\,.\,x^2$, whence there is

$$d\delta = \mathrm{K}\,.\,\delta^2 x^2.$$

The quantity $d\delta$ is the longitudinal aberration, and its value is obtained by substituting in this expression the value of K, as given by the preceding articles.

Let A*f* be the extreme ray, meeting the axis in *f*, and the perpendicular to the axis, FO, erected at F the geometric focus, in O; then F*f* is the longitudinal aberration, and FO the lateral aberration, and by similar triangles

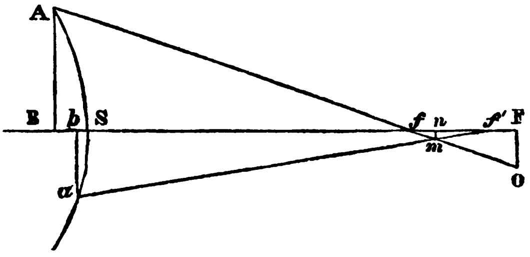
$$\mathrm{FO} = \mathrm{F}f\,.\,\frac{\mathrm{AB}}{\mathrm{B}f} = \mathrm{F}f\,.\,\frac{x'}{\delta},\ \text{nearly};$$

x' denoting the semi-aperture of the last lens. Now, if α and β denote the reciprocals of the distances of the foci of incident and refracted rays for the first lens, α' and β' the analogous quantities for the second, &c. there is

$$x' = \frac{\beta\beta'\beta''\ \&\text{c.}}{\alpha'\alpha''\alpha'''\ \&\text{c.}}\,.\,x = ax,$$

a denoting the ratio $\dfrac{\beta\beta'\beta''\ \&\text{c.}}{\alpha'\alpha''\alpha'''\ \&\text{c.}}$. Accordingly, if we substitute for F*f* and x' their values in the expression of FO, we have

$$\text{lateral aberration} = a\mathrm{K}\delta x^3.$$



(189.) To determine the least circle of aberration, let A*f* be the extreme ray meeting the axis in *f*; *af'* any ray at the other side of the axis, meeting it in *f'*, and *mn* the perpendicular let fall from their intersection upon the axis. Then, reasoning as in (65.), it will be evident that mn, when a *maximum*, will be the radius of the least circle of aberration.

To find its value, let x denote the semi-aperture of the first lens corresponding to AS, x' that corresponding to as; then, F being the geometric focus, there is

$$\mathrm{F}f = \mathrm{K}\delta^2x^2, \quad \mathrm{F}f' = \mathrm{K}\delta^2x'^2, \text{ and subtracting,}$$

$$ff' = \mathrm{K}\delta^2(x^2 - x'^2).$$

Again, on account of similar triangles, we have

$$nf = mn\,.\,\frac{\mathrm{S}f}{\mathrm{AS}} = \frac{\rho.\delta}{ax},\ q\,.\,p,$$

$$nf' = mn\,.\,\frac{\mathrm{S}f'}{as} = \frac{\rho.\delta}{ax'},\ q\,.\,p,$$

mn being denoted by ρ. Wherefore, adding, there is

$$ff' = \frac{\rho.\delta}{a}\left(\frac{1}{x} + \frac{1}{x'}\right).$$

Finally, equating these two values of ff', we obtain

$$\rho = a\mathrm{K}\delta(x - x')xx'.$$

But x' being the variable in this expression, ρ varies as $(x - x')x'$, and is a maximum when the latter is so; that is, when $x' = \frac{1}{2}x$: wherefore, substituting this value of x' in the expression of ρ, its maximum value, or the radius of the least circle of aberration, is

$$\rho = \tfrac{1}{4}a\mathrm{K}\delta x^3.$$

And comparing this with the result of the preceding article, we learn that in any combination of lenses whatever, the radius of the least circle of aberration is one-fourth of the lateral aberration of the extreme ray.

With respect to the position of the centre of this circle, we have already found

$$nf = \frac{\rho.\delta}{ax} = \tfrac{1}{4}\mathrm{K}\delta^2x^2, \quad \text{but } \mathrm{F}f = \mathrm{K}\delta^2x^2,$$

wherefore, subtracting, there is

$$n\mathrm{F} = \tfrac{3}{4}\mathrm{K}\delta^2x^2;$$

wherefore the distance of the centre of the circle from the geometric focus is three-fourths of the longitudinal aberration of the extreme ray.

(190.) When the lenses composing the system are *in contact*, $\alpha' = \beta$, $\alpha'' = \beta'$, &c.; and therefore

$$a = \frac{\beta\beta'\beta''\ \&c.}{\alpha'\alpha''\alpha'''\ \&c.} = 1.$$

Accordingly, the expression of the radius of the least circle of aberration, in this case, becomes

$$\rho = \tfrac{1}{4}\mathrm{K}\delta x^3.$$

In which case also $\mathrm{K} = \kappa + \kappa' + \kappa'' + \&c.$ (183.); $\kappa, \kappa', \kappa''$, &c. being the coefficients of the square of the semi-aperture in the values of $d\beta$ for the several lenses composing the system.

Thus, for example, if the rays are incident parallel upon the plane side of a plano-spherical lens, there is

$$\mathrm{K} = \kappa = \frac{\mu^2(\mu-1)}{2r^3}\ (172.),\ \text{and}\ \delta = f = \frac{r}{\mu-1}\ (157.);$$

$$\therefore \rho = \tfrac{1}{8}\frac{\mu^2x^3}{r^2},\quad \text{or}\ 2\rho = \frac{\mu^2x^3}{(2r)^2}.$$

When parallel rays are incident upon a lens of *best form*,

$$\mathrm{K} = \kappa = \frac{\mu(\mu-\frac{1}{4})}{2(\mu+2)(\mu-1)^2}\varphi^3,\ (175.)\ \text{and}\ \delta = \frac{1}{\varphi};$$

$$\therefore \rho = \frac{\mu(\mu-\frac{1}{4})}{8(\mu+2)(\mu-1)^2}\varphi^2x^3.$$

V.

Of Images formed by Refraction at spherical Surfaces.

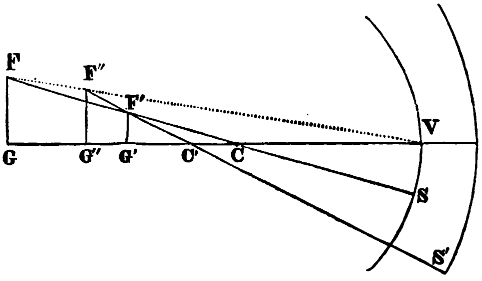
(191.) Of images formed by refraction at a single spherical surface little need be said, inasmuch as such surfaces are never found in practice except in combination with a second, whether plane or spherical, in the form of a lens. From what has been already said of images in general, it will readily appear that, when an object of any form is presented to a spherical refracting surface, the points of the image corresponding to the several points of the object are found by drawing from the latter lines to the centre of the refracting surface, and taking on these lines points whose distances from the centre are calculated by the formula

$$u' - \varrho = m(u - \varrho),$$

in which u represents the reciprocal of the distance of any point of the object from the centre, and u' that of the corresponding point of the image (145.).

By an examination of this formula it will be easily seen that the image of a spherical surface concentric with the refracting surface is also a spherical surface having the same centre; that the image of a plane is the surface generated by the revolution of a conic section round its axis; and that the linear magnitudes of the object and image, supposing them to be spherical surfaces concentric with the refractor, are as their distances from the centre.

(192.) When there is a second refracting surface, the focus of the doubly refracted pencil, or the point of the image corresponding to any point of the object, will be determined in the following manner:



C being the centre of the first surface, and C' that of the second, the line C'CV, passing through them, will be the *common axis* of the two surfaces. Let F be any focus of incident rays situated out of this axis, and FCS the line drawn from it through the centre of the first surface, and meeting that surface in S;

that line will be the axis of the pencil of rays incident from the point F upon the first surface: and if the distance SF' be taken, calculated according to the formula (136.), the point F' will be the focus of rays refracted by the first surface, and therefore the focus of rays incident upon the second. Through this point, therefore, let the line $F'C'S'$ be drawn through the centre of the second surface, and in this line let the distance $S'F''$ be taken, calculated as before; then F'' will be the focus of the pencil after refraction by both surfaces.

It is obvious that the same method may be extended to any number of refracting surfaces.

Now, confining our attention to the case of two surfaces, it is easy to see that, when the distance between these surfaces is inconsiderable, and the obliquity of the incident pencil, or the inclination of its axis to the common axis of the two surfaces, very small, $F'S$ and $F'S'$, the distances of the focus of the pencil after the first refraction from the two surfaces will be *quam proximè* equal. It follows therefore that, in this case, the same relation will subsist between FS and $F''S'$, the distances of the foci of the incident and emergent rays from the surface, as when those points are situated on the common axis of the two surfaces; or, in other words, that the distance of the focus of the doubly refracted pencil from the lens will be the same as when the pencil is incident perpendicularly.

To find the distance of the focus of refracted rays from the axis: on account of similar triangles, there is

$$\frac{FG}{F'G'} = \frac{FC}{F'C} = \mu . \frac{FS}{F'S}, \quad \frac{F'G'}{F''G''} = \frac{F'C'}{F''C'} = \frac{1}{\mu} . \frac{F'S'}{F''S'};$$

multiplying these equations, and observing that $F'S' = F'S$,

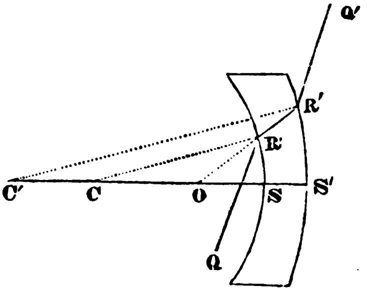
$$\frac{FG}{F''G''} = \frac{FS}{F''S'} = \frac{FV}{F''V}, \quad q.p.$$

From this it follows that the points F, F'', and V, are in the same right line, or, that the focus of refracted rays lies in the right line joining the focus of incident rays with the vertex of the lens. This right line, therefore, may be regarded as the axis of the doubly refracted pencil, and the position of the focus of the refracted rays on it will be determined by the known formula for central rays (155.).

(193.) We are led to the conclusion of the preceding article also from the consideration of the ray which suffers no deviation in passing through a lens.

When a ray of light passes in any manner through a lens of any thickness, it is evident that the incident and emergent portions of the ray will be parallel, or the deviation nothing, when the tangents at the points of incidence and emergence (and, therefore, also the radii drawn to these points) are parallel; for, in this case, the ray is under the same circumstances as if it had passed through a medium bounded by parallel planes.

Let QRR'Q' be the course of such a ray, of which the incident and emergent portions, QR and Q'R', are parallel; and, C and C' being the centres of the first and second surface respectively, let the radii, CR, C'R', be drawn to the points of incidence and emergence; also, let the portion of the ray within the lens, RR', be produced to meet the axis in O. Then, on account of the parallelism of the radii, CR and C'R', the triangles, COR, C'OR', are similar, and



$$\text{C}'\text{O} = \text{CO} \cdot \frac{\text{C}'\text{R}'}{\text{CR}}, \quad \text{whence CC}' = \text{CO}\left(\frac{\text{C}'\text{R}'}{\text{CR}} - 1\right);$$

whence, denoting the radii of the surfaces, CR and C'R', by r and r', there is

$$\text{CO} = \text{CC}' \cdot \frac{r}{r' - r}.$$

From which it appears that the point O, in which the portion of the ray within the lens meets the axis, is invariable. This point is called the *centre of the lens.*

To compute the distance of this point from the surface:

$$\text{OS} = \text{CS} - \text{CO} = r \cdot \frac{r' - r - \text{CC}'}{r' - r};$$

but $r' - r - \text{cc}' = \text{ss}'$, the thickness of the lens; wherefore, if the thickness be denoted by δ, there is

$$\text{os} = \delta \cdot \frac{r}{r' - r}.$$

From which we learn that the distance of the centre of the lens from the first surface is to the thickness of the lens, as the radius of the first surface is to the difference of the radii of the two surfaces.

The preceding expression has been calculated for the case of the *concavo-convex* lens, in which the radii, r and r', are both *positive.* It is obvious, however, that the formula includes all cases, if we observe only that the quantities, r, r', and os, are to be considered as affected with the positive or negative sign, according as they lie from the surface *towards the incident light*, or *from* it (137.).

In the *concavo-convex* lens, having its more curved surface turned towards the incident light, r and r' are both positive: in the *meniscus* they are both negative; and the value of os in each case is the same, namely, $\delta \cdot \frac{r}{r' - r}$. Wherefore, since r' is greater than r, the value of os is *positive;* and therefore the centre of the lens lies without it and at the side of the more curved surface. It is obvious, that if the lens be turned in the opposite direction, the position of the centre will not be altered.

In the *double convex* lens r is negative and r' positive; in the *double concave* their signs are the opposite; and accordingly the value of os is the same in both cases, namely, $-\delta \cdot \frac{r}{r' + r}$. Wherefore, in these lenses, os is negative and less than δ; and, accordingly, the centre of the lens lies within it, bisecting its thickness when the surfaces are of equal curvatures.

In the *plano-convex* and *plano-concave* lenses, if the curved surface be turned towards the incident light, the radius of the second surface, r', is infinite, and os $= 0$. The centre of the lens, therefore, coincides with the vertex of the spherical surface.

Finally, in a lens whose thickness is inconsiderable, os becomes indefinitely small, and the centre of the lens coincides *q.p.* with the surface. And, if the obliquity of the ray passing through this centre be small, the incident and emergent portions may be regarded as parts of the same right line; the portion of the ray within the lens being, in this case, inconsiderable. This ray, therefore, may be regarded as undergoing no refraction whatever.

(194.) From what has just preceded, it appears that when a pencil of rays, whose obliquity to the axis is small, is incident upon a thin lens, there is one ray of the incident pencil, that, namely, which passes through the centre of the lens, which may be regarded as undergoing no refraction whatever. This ray is called the *principal ray* of the oblique pencil, and is to be considered as its axis. Wherefore the focus of the refracted pencil will be obtained by joining the focus of the incident rays with the centre of the lens, and taking on this line a point whose distance from the lens is calculated by the formula for central rays,

$$\alpha' = \alpha + \varphi.$$

From which it will readily appear that the form of the image of any object, produced by such a lens and referred to its centre, will be the same in species as that produced by reflexion at a spherical surface, and referred to its centre.

Thus, when the object is a portion of a *spherical surface*, whose centre is the centre of the lens, α, the reciprocal of the distance of the several points of the object from the centre, is constant; α', therefore, is likewise constant, and the form of the image will be also a spherical surface having the same centre, and whose radius is the reciprocal of α', or of $\alpha + \varphi$.

Again, if the object be a *plane* perpendicular to the axis of the lens, we may confine our attention to the section of this plane formed by any plane passing through the axis. If, then, a denote the reciprocal of the portion of the axis intercepted between the lens and this section, and θ the angle contained, with the axis, by the line drawn from any point of the object to the centre of the lens,

$$\alpha = a.\cos.\theta, \text{ and } \alpha' = \phi + a.\cos.\theta.$$

From this it follows, as in (70.), that the section of the image is a *conic section*, whose *axis* is the axis of the lens, and *focus* its centre; and that the *principal parameter* of the section is double the focal length of the lens, and its *excentricity* equal to the ratio of the focal length to the perpendicular distance of the object from the lens. The section is therefore an *ellipse*, *hyperbola*, or *parabola*, according as the distance of the object from the lens is greater, less than, or equal to, the principal focal length. When the object is infinitely distant, the ellipse becomes a *circle*, whose centre is that of the lens, and radius its focal length. When the object coincides with the lens, the hyperbola becomes a *right line* coincident with the object.

(195.) When the section of the object is perpendicular to the axis of the lens, and subtends a small angle at its centre, it may, without sensible error, be considered as a circular arc whose centre is the centre of the lens. And the section of the image being, in this case, a circular arc having the same centre, and subtending the same angle at that centre, it is obvious that the linear magnitudes of the object and image will be as the radii of these circles, or as their distances from the centre of the lens. Wherefore, if the distances of the object and image from the centre of the lens be denoted by δ and δ', and their linear magnitudes by m and m',

$$\frac{m'}{m} = \frac{\delta'}{\delta} = \frac{f}{\delta + f}.$$

When the lens is of the *concave* kind, f is *positive*; and, since the incident rays are always *divergent* if the object presented to the lens be a *real* object, δ is always positive, and accordingly the value of the ratio $\frac{m'}{m}$, in this case, is always less than unity; or, the image of an object formed by a concave lens is always less than it; the ratio decreasing from unity to nothing, as the object recedes from the lens to an infinite distance.

When the lens is of the *convex* kind, f is *negative*, and

$$\frac{m'}{m} = \frac{f}{f-\delta}.$$

Hence, when the object coincides with the lens, $\delta = 0$, and the ratio $\frac{m'}{m}$ is equal to unity, or the image equal to the object. As the distance of the object increases, the value of the fraction $\frac{m'}{m}$ increases indefinitely, until, when $\delta = f$, the ratio becomes infinite; that is, when the object arrives at the principal focus the image is infinitely great compared with it. When the object is beyond the principal focus, $\delta > f$, and the ratio $\frac{m'}{m}$ becomes negative: its value also diminishes indefinitely as the distance of the object increases, becoming equal to -1, when $\delta = 2f$, and vanishing altogether when δ is infinite. Hence, when the distance of the object from the lens is double its focal length, the image and object are again equal; and when the object is infinitely distant, the image is infinitely small in comparison with it.

Since the axes of the several pencils intersect at the centre of the lens, it is obvious that the image will be *erect* with respect to the object, when they lie at the same side of the lens, *i. e.* when δ and δ' are affected with the same sign; it will be *inverted* when they are at opposite sides, or δ and δ' affected with opposite signs. Hence it is evident that the position of the image with respect to the object will be determined by the sign of the fraction $\frac{m'}{m}$, which is equal to $\frac{\delta'}{\delta}$, being *erect* when that fraction is *positive*, *inverted* when it is *negative*. It appears, therefore, from what has been said above, that in a lens of the concave kind the image is always erect with respect to the object; while the image formed by a convex lens will be erect only when the object between the lens and the principal focus, and in all other s inverted.

It is evident, also, that the inverted image is always *real*, the rays actually meeting there, and the erect *imaginary*.

(196.) In any combination of lenses the image formed by any one lens of the system is to be considered as the object presented to the next, &c., and thus the position and magnitude of the last image will be computed on the principles already established.

Let α and β denote the reciprocals of the distances of the object and its image produced by the first lens of the system, α' and β' the analogous quantities for the second lens, α'' and β'' for the third, &c. Also, let m denote the linear magnitude of the object; m', m'', m''', &c. $m^{(n)}$, those of the images formed by the 1st, 2d, 3d, &c. and nth lens, respectively; then, from what has been said, it will appear that

$$\frac{m'}{m} = \frac{\alpha}{\beta}, \quad \frac{m''}{m'} = \frac{\alpha'}{\beta'}, \quad \frac{m'''}{m''} = \frac{\alpha''}{\beta''}, \text{ \&c.}$$

And multiplying these equations together, we find

$$\frac{m^{(n)}}{m} = \frac{\alpha\alpha'\alpha''\ldots\alpha^{(n-1)}}{\beta\beta'\beta''\ldots\beta^{(n-1)}},$$

an equation determining the ratio of the linear magnitudes of the object and its last image.

The relations amongst the quantities α, β, α', β', &c. which enter this expression are given by the equations of (168.), by means of which they may be all determined when the first is given.

When the lenses composing the system are in contact, $\beta = \alpha'$, $\beta' = \alpha''$, $\beta'' = \alpha'''$, &c.; wherefore, if the reciprocal of the last distance, $\beta^{(n-1)}$, be denoted by $\alpha^{(n)}$, the expression of the ratio becomes

$$\frac{m^{(n)}}{m} = \frac{\alpha}{\alpha^{(n)}}.$$

The relation between the quantities $\alpha^{(n)}$ and α is given by the equation

$$\alpha^{(n)} - \alpha = \varphi,$$

in which φ denotes the power of the system, or the sum of the powers of the component lenses (167.); and if we substitute

When the lens is of the *convex* kind, f is *negative*, and

$$\frac{m'}{m}=\frac{f}{f-\delta}.$$

Hence, when the object coincides with the lens, $\delta = 0$, and the ratio $\frac{m'}{m}$ is equal to unity, or the image equal to the object. As the distance of the object increases, the value of the fraction $\frac{m'}{m}$ increases indefinitely, until, when $\delta = f$, the ratio becomes infinite; that is, when the object arrives at the principal focus the image is infinitely great compared with it. When the object is beyond the principal focus, $\delta > f$, and the ratio $\frac{m'}{m}$ becomes negative: its value also diminishes indefinitely as the distance of the object increases, becoming equal to -1, when $\delta = 2f$, and vanishing altogether when δ is infinite. Hence, when the distance of the object from the lens is double its focal length, the image and object are again equal; and when the object is infinitely distant, the image is infinitely small in comparison with it.

Since the axes of the several pencils intersect at the centre of the lens, it is obvious that the image will be *erect* with respect to the object, when they lie at the same side of the lens, *i. e.* when δ and δ' are affected with the same sign; it will be *inverted* when they are at opposite sides, or δ and δ' affected with opposite signs. Hence it is evident that the position of the image with respect to the object will be determined by the sign of the fraction $\frac{m'}{m}$, which is equal to $\frac{\delta'}{\delta}$, being *erect* when that fraction is *positive*, *inverted* when it is *negative*. It appears, therefore, from what has been said above, that in a lens of the concave kind the image is always erect with respect to the object; while the image formed by a convex lens will be erect only when the object is between the lens and the principal focus, and in all other cases inverted.

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And multiplying these equations together, we find

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in which φ denotes the power of the system, or the sum of the powers of the component lenses (167.); and if we substitute

When the lens is of the *convex* kind, f is *negative*, and

$$\frac{m'}{m} = \frac{f}{f-\delta}.$$

Hence, when the object coincides with the lens, $\delta = 0$, and the ratio $\frac{m'}{m}$ is equal to unity, or the image equal to the object. As the distance of the object increases, the value of the fraction $\frac{m'}{m}$ increases indefinitely, until, when $\delta = f$, the ratio becomes infinite; that is, when the object arrives at the principal focus the image is infinitely great compared with it. When the object is beyond the principal focus, $\delta > f$, and the ratio $\frac{m'}{m}$ becomes negative: its value also diminishes indefinitely as the distance of the object increases, becoming equal to -1, when $\delta = 2f$, and vanishing altogether when δ is infinite. Hence, when the distance of the object from the lens is double its focal length, the image and object are again equal; and when the object is infinitely distant, the image is infinitely small in comparison with it.

Since the axes of the several pencils intersect at the centre of the lens, it is obvious that the image will be *erect* with respect to the object, when they lie at the same side of the lens, *i. e.* when δ and δ' are affected with the same sign; it will be *inverted* when they are at opposite sides, or δ and δ' affected with opposite signs. Hence it is evident that the position of the image with respect to the object will be determined by the sign of the fraction $\frac{m'}{m}$, which is equal to $\frac{\delta'}{\delta}$, being *erect* when that fraction is *positive*, *inverted* when it is *negative*. It appears, therefore, from what has been said above, that in a lens of the concave kind the image is always erect with respect to the object; while the image formed by a convex lens will be erect only when the object is between the lens and the principal focus, and in all other cases inverted.

It is evident, also, that the inverted image is always *real*, the rays actually meeting there, and the erect *imaginary*.

(196.) In any combination of lenses the image formed by any one lens of the system is to be considered as the object presented to the next, &c., and thus the position and magnitude of the last image will be computed on the principles already established.

Let α and β denote the reciprocals of the distances of the object and its image produced by the first lens of the system, α' and β' the analogous quantities for the second lens, α'' and β'' for the third, &c. Also, let m denote the linear magnitude of the object; m', m'', m''', &c. $m^{(n)}$, those of the images formed by the 1st, 2d, 3d, &c. and nth lens, respectively; then, from what has been said, it will appear that

$$\frac{m'}{m} = \frac{\alpha}{\beta}, \quad \frac{m''}{m'} = \frac{\alpha'}{\beta'}, \quad \frac{m'''}{m''} = \frac{\alpha''}{\beta''}, \text{ \&c.}$$

And multiplying these equations together, we find

$$\frac{m^{(n)}}{m} = \frac{\alpha\alpha'\alpha''\ldots\alpha^{(n-1)}}{\beta\beta'\beta''\ldots\beta^{(n-1)}},$$

an equation determining the ratio of the linear magnitudes of the object and its last image.

The relations amongst the quantities α, β, α', β', &c. which enter this expression are given by the equations of (168.), by means of which they may be all determined when the first is given.

When the lenses composing the system are in contact, $\beta = \alpha'$, $\beta' = \alpha''$, $\beta'' = \alpha'''$, &c.; wherefore, if the reciprocal of the last distance, $\beta^{(n-1)}$, be denoted by $\alpha^{(n)}$, the expression of the ratio becomes

$$\frac{m^{(n)}}{m} = \frac{\alpha}{\alpha^{(n)}}.$$

The relation between the quantities $\alpha^{(n)}$ and α is given by the equation

$$\alpha^{(n)} - \alpha = \varphi,$$

in which φ denotes the power of the system, or the sum of the powers of the component lenses (167.); and if we substitute

the value of $\alpha^{(n)}$, thus obtained, in the expression of the ratio it will be

$$\frac{m^{(n)}}{m} = \frac{\alpha}{\alpha+\varphi};$$

from which it appears that the ratio of the linear magnitudes of the object and its image, formed by any combination of lenses in contact, is the same as for a single lens whose power is equal to the power of the system.

(197.) Let it be required to determine the brightness of the image of any luminous object formed by a lens or speculum.

Let m and m' denote the linear magnitudes of the object and image, δ and δ' their distances from the lens, or centre of the speculum, and A the linear semi-aperture. Then, if the angle subtended by the latter quantity at the luminous object be very small, the surface of the lens may be considered as a portion of the hemisphere whose centre is the luminous object and radius its distance from the lens; and the quantity of light incident upon it will be to that incident upon the entire hemisphere in the ratio of their areas, *i. e.* as $\pi A^2 : 2\pi\delta^2$. Wherefore, if the quantity of light incident upon the entire hemisphere be denoted by Q, the portion incident on the lens or speculum will be $Q \cdot \frac{A^2}{2\delta^2}$. And, if the quantity of the transmitted or reflected light be to that of the incident light in the ratio of $\varrho : 1$, the quantity of light in the image will be $\varrho Q \frac{A^2}{2\delta^2}$. Accordingly, if this be denoted by Q', there is

$$\frac{Q'}{Q} = \tfrac{1}{2}\varrho\left(\frac{A}{\delta}\right)^2.$$

Hence the absolute quantity of light in the image varies as the apparent magnitude of the lens, as seen from the object; and, when the distance of the object from the lens is given, simply as the square of the aperture. This is what the eye estimates when the image has no sensible magnitude, as in the case of the fixed stars. Hence the importance of a large object-glass in sidereal observations.

The densities of the light in the object and image are as the absolute quantities of light directly and inversely as their areas, or the squares of their linear magnitudes. Wherefore, if these densities be denoted by D and D′,

$$\frac{\mathrm{D}'}{\mathrm{D}} = \frac{\mathrm{Q}'}{\mathrm{Q}} \cdot \left(\frac{m}{m'}\right)^2 = \tfrac{1}{2}\rho\left(\frac{\mathrm{A}}{\delta'}\right)^2,$$

substituting for $\frac{\mathrm{Q}'}{\mathrm{Q}}$ and $\frac{m}{m'}$, their values $\frac{1}{2}\varrho\left(\frac{\mathrm{A}}{\delta}\right)^2$ and $\frac{\delta}{\delta'}$.

The brightness of the image is measured by the density of the light in it, and therefore varies as the apparent magnitude of the lens, as seen from the image, whatever be the distance of the object. Hence the density of the sun's light in the focus of a lens or speculum varies as the square of the linear aperture directly and inversely as the square of the focal length.

From the value of the ratio $\frac{\mathrm{D}'}{\mathrm{D}}$ it appears that the density of the light, or the degree of illumination of the image, is much less than that of the object, even supposing that there is no light lost in reflexion or refraction, or that $\varrho = 1$.

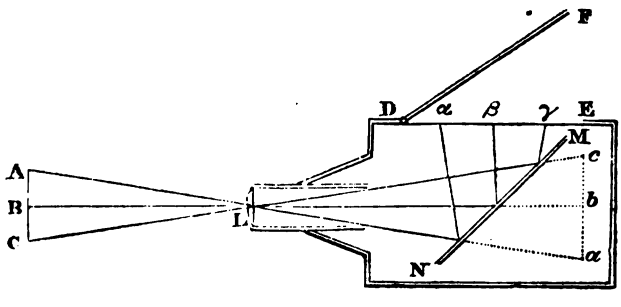
(198.) In what has preceded respecting images it has been supposed, that the pencils of rays diverging from each point of the object are reflected or refracted accurately to a point. This however does not take place in general in any of the cases that have been examined, except in that of reflexion by a plane surface. In all other cases the focus of the reflected or refracted rays, corresponding to each point in the object, will be, not a mathematical point, but a physical point, or small circle, over which the rays are diffused; and as these circles overlay one another, there will thereby be produced a confusion in the image proportional to their magnitude. These circles, we have seen, are the least circles of aberration or diffusion; and hence the importance of correcting or diminishing the aberration in lenses used in optical instruments.

(199.) The preceding theory is exemplified in many interesting and useful applications.

If the light be admitted into a darkened chamber through a circular aperture made in the window-shutter, and in this aperture a convex lens be placed, it is evident, from what has been said, that inverted images of external objects will be formed within the room at their proper distances from the lens; and that if the objects be at a considerable distance from the lens compared with its focal length, the distances of their images will be very nearly the same, and equal to the focal length of the lens. Hence if a sheet of paper be placed perpendicularly to the axis of the lens, at a distance equal to its focal length, an inverted picture of the external scene will be formed there, whose brightness, *cæt. par.*, will vary in the duplicate ratio of the angle which the diameter of the lens subtends at its principal focus.

In order that the lens may be directed to different external objects, it is adapted to a *scioptric ball.* The instrument so called is a solid sphere of polished wood fitting in a hollow frame of the same, whose interior surface is a portion of a spherical surface of the same diameter as the ball which it contains, so that the latter may revolve within it every way. Through the centre of the ball a cylindrical hole is cut, at the extremity of which the lens is adapted perpendicularly to the axis of the hole, so that the latter coincides with the axis of the lens. It is evident that by this arrangement the axis of the lens may be directed to any object within view.

For the purposes of drawing, it is convenient that the image should be thrown into a horizontal position. This is effected by placing a plane mirror between the lens and its principal focus, and inclined to the axis of the lens at an angle of 45°; a second image will thus be formed before the mirror, at the same distance as that which the rays tend to form behind it, and the plane of the former image is perpendicular to that of the latter (44), and therefore horizontal.



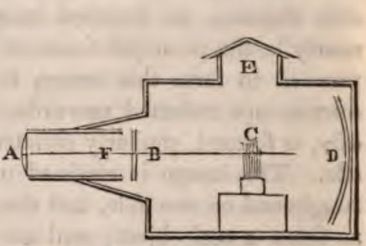
(200.) Such is the principle of the portable *camera obscura;* a box from which all extraneous light is perfectly excluded, being substituted for the darkened chamber. This instrument is represented in the adjoining figure: L is a lens fitted in a sliding tube, and MN a plane mirror inclined at an angle of 45° to its axis L*b*; now, if ABC be any external object at a considerable distance, an inverted image of it, *abc*, will be formed nearly in the principal focus of the lens; but the rays which proceed to form this image, being intercepted by the plane mirror, are reflected upwards, and thus a horizontal image, $\alpha\beta\gamma$, is formed, similarly situated with respect to the mirror as *abc.* This image is received upon a piece of plane glass, DE, roughened on one side, and thus the rays will diverge from it as from a real object; and the extraneous light is excluded from the picture by means of a lid, DF, with a curtain attached, which covers the head of the spectator.

In this construction of the instrument, the spectator having his face turned towards the object, it is evident that the image will appear erect as to top and bottom; but that with respect to right and left it will be inverted, the rays coming from the right side of the object crossing the axis of the lens at its centre, and proceeding to the left of the image, and *v. v.*

In the construction just described, the lower part of the mirror forms an angle of 45° with the axis of the lens: if this position be reversed, and the *upper* part be inclined to the axis at the same angle, it is evident that the reflected image will be thrown *downwards*, and may be received upon a table placed at the proper distance. In this arrangement the spectator has his back turned towards the object, and views the image through an opening in the front of the box, or, as is most usual, is himself enclosed within the chamber by means of a curtain covering his person, as well as the table on which the image is thrown; and it is evident that, the spectator having his back to the object, the position of the image, with respect to right and left, as well as with respect to top and bottom, will be the same as that of the object.

This construction, which is generally that of the larger instruments of this kind, is the best adapted to the purposes of drawing. It is frequently modified by placing the plane mirror so as to receive the rays and bend them into the vertical before they meet with the lens, which must therefore be horizontal. In this arrangement the whole focal length of the lens lies in the vertical, and therefore the frame of the instrument is necessarily taller than in the other cases; in all other respects it is the same.

(201.) In the *magic lantern* the object, instead of being at a considerable distance, is placed near the focus of the lens, and thus a distant and magnified image of it is formed upon a screen placed to receive it.



The adjoining figure represents the lantern, in the front of which is a convex lens, A, fitted in the extremity of a sliding tube. B is a rectangular aperture or slit into which are introduced the objects to be represented, which are usually grotesque figures painted in transparent colours upon glass plates or *slides*. C is a lamp, in the centre of the box, by which the object is illuminated. D is a reflector behind the lamp to concentrate its light upon the object, and E is a chimney above it with a projecting roof to intercept the light. Then, the room being darkened, one of the slides is introduced at B in an *inverted* position, and the tube containing the lens is moved until the distance of the lens from the slide, AB, is a little greater than its focal length, AF. This being done, it is evident that a magnified image of the object on the slide will be formed at a considerable distance from the lens, and may be received upon a screen placed at the proper distance. This image will be inverted with respect to the object on the slide, and therefore erect with respect to the spectator.

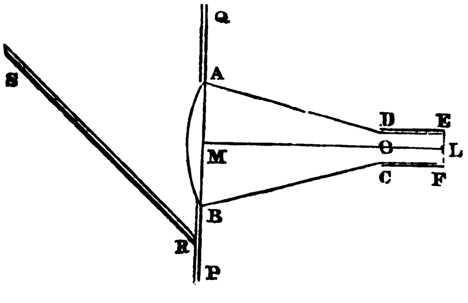
If the lantern and screen be fixed, the adjustment is to be performed by moving the sliding tube which contains the lens, and will be always possible, provided the distance of the screen from the slide be greater than four times the focal length of the lens (159.).

The magnifying power of this instrument, or the ratio $\frac{m'}{m}$, is equal to $\frac{f}{\delta - f}$, the lens being convex, or to $\frac{AF}{BF}$. Hence, in the same instrument, the magnifying power varies inversely as BF, the distance of the slide from the principal focus of the lens. The brightness of the image varies as $\left(\frac{A}{\delta'}\right)^2$, and therefore in the same instrument is inversely as the square of the distance of the image from the lens.

In the common magic lantern the image is reflected by the screen to the spectator, who is therefore at the same side of it with the lantern itself. But in the *phantasmagoria* the image is received upon a thin transparent screen, placed between the spectator and the lantern, and the magnitude of the image is made to vary by a simultaneous motion of the lantern and sliding tube, which are so regulated that the image may always fall upon the screen. If the brightness of the image increased and decreased with its size, it would bear all the appearance of an object advancing and retiring; this, however, is the reverse of what takes place under ordinary circumstances, and, to effect it, it is necessary to have some means of modifying the quantity of light, so as to diminish it when the lantern is brought near to the screen, and *v. v.* When these various contrivances are well arranged, the appearances produced are in the highest degree entertaining and deceptive.

(202.) The principle of the *solar microscope* is the same as that of the magic lantern; the object whose image is to be represented being in this case illuminated by a beam of the sun's light, which is thrown into the axis of the tube by the aid of reflectors.

ABCD represents a conical tube, the less extremity of which, CD, is cylindrical and furnished with a sliding tube, CDEF, which fits it exactly. In the extremity of the sliding tube is



a convex lens, L, of small aperture, by which the image of the object is to be formed; and at the wider extremity of the conical tube is a broad convex lens, AB, the use of which is to concentrate the rays of the sun upon the object, and thus to illuminate it strongly*. This extremity of the instrument is inserted in a square frame, PQ, by means of which it may be attached to a window-shutter at an aperture corresponding to the size of the lens, AB. To this frame is attached also, at the other side, a plane mirror, RS, by which the light of the sun may be thrown into the instrument. This mirror is moveable round a hinge at R, by which means its inclination to the plane PQ, or to the axis of the instrument, may be altered at pleasure; and the hinge itself is capable of a rotatory movement round the axis ML, which enables the observer to vary at will the plane of reflexion of the mirror.

When the instrument is used, it is attached to the aperture in the shutter of a darkened room, and a beam of the sun's light thrown into it, in the direction of its axis, by the adjustments of the plane mirror. The object to be examined, which is generally some minute and partly transparent natural object, is then introduced into the axis of the tube at O, through an aperture in the side; the point, O, at which the object is placed being near the principal focus of the illuminating lens, but not exactly at that point, on account of the intensity of the heat there. The sliding tube, CDEF, is then moved by means of a rack and screw until the distance of the lens from the object, LO, is a little greater than the focal length of the lens. It is manifest, then, that a magnified image of the object upon a bright ground will be formed on a screen placed at a proper distance.

The magnifying power in this instrument is the same as in the magic lantern, being equal to the focal length of the lens divided by the distance of the object from its principal focus. Instruments of this kind are usually made with a very high magnifying power.

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* A second and smaller lens is generally added, near the principal focus of the first, for the purpose of concentrating still further the admitted beam.

CHAPTER VII.

OF LIGHT REFRACTED AT ANY CURVED SURFACES.

I.

General Theory of Refraction at any curved Surfaces.

(203.) LET the angles which the incident and refracted rays make with the axis of abscissæ be denoted by ω and ω'; then, as in the case of reflexion (74.), the cosines of the angles which these rays form with the tangent to the curve at the point of incidence are, respectively,

$$\cos.\omega\,.\frac{dx}{ds}+\sin.\omega\,.\frac{dy}{ds},$$

$$\cos.\omega'.\frac{dx}{ds}+\sin.\omega'.\frac{dy}{ds}.$$

Now, these cosines are $\pm$ sin.(inc.) and $\mp$ sin.(ref.) respectively; therefore, substituting these values in the equation sin.(inc.) $=m\,.$ sin.(ref.), we have

$$\cos.\omega\,.\,dx+\sin.\omega\,.\,dy+m(\cos.\omega'.\,dx+\sin.\omega'.\,dy)=0;$$

an equation which determines ω', the angle which the refracted ray makes with the axis, when ω, the angle which the incident ray makes with the same, is given.

(204.) Let (α,β) be the co-ordinates of any point of the incident ray, (α',β') those of the refracted ray; then, as these rays pass through the point of incidence, whose co-ordinates are (x,y), their equations are

$$\beta - y = \tan.\omega(\alpha - x),$$
$$\beta' - y = \tan.\omega'(\alpha' - x).$$

And, if we eliminate ω and ω' between these equations and that of the preceding article, as has been done in the case of reflected light, the resulting equation will express the relation between (α, β), (α', β'), and the co-ordinates of the refracting curve. Accordingly, when the co-ordinates (α, β) are determined by the condition to which the incident light is subject, this equation exhibits the relation between α' and β', the co-ordinates of the refracted ray, for each point of the refracting curve.

To proceed with this elimination:—from the equations just obtained, we deduce, as before,

$$\cos.\omega = \frac{\alpha - x}{\varrho}, \quad \sin.\omega = \frac{\beta - y}{\varrho},$$
$$\cos.\omega' = \frac{\alpha' - x}{\varrho'}, \quad \sin.\omega' = \frac{\beta' - y}{\varrho'},$$

in which we have made, for abbreviation,

$$\varrho = \sqrt{(\alpha - x)^2 + (\beta - y)^2},$$
$$\varrho' = \sqrt{(\alpha' - x)^2 + (\beta' - y)^2}.$$

And these values being substituted in equation of preceding article, it becomes

$$\frac{(\alpha - x)dx + (\beta - y)dy}{\varrho} + m.\frac{(\alpha' - x)dx + (\beta' - y)dy}{\varrho'} = 0;$$

an equation which gives the relation between α' and β', the co-ordinates of the refracted ray, when α and β, the co-ordinates of the incident ray, are known.

When the incident ray is parallel to the axis, $\beta - y = 0$; wherefore $\varrho = \alpha - x$, and the preceding equation is reduced to

$$\varrho' dx + m\left[(\alpha' - x)dx + (\beta' - y)dy\right] = 0.$$

(205.) If the values of ϱ^2 and ϱ'^2 be differentiated relatively to x and y only, there is

$$(\alpha - x)dx + (\beta - y)dy = - \rho d\rho$$
$$(\alpha' - x)dx + (\beta' - y)dy = - \rho' d\rho';$$

and if these values be substituted in the equation of the preceding article, it becomes

$$d\rho + md\rho' = 0;$$

from which we learn that the function $(\rho + m\rho')$ is a *minimum*, when taken from any assumed point in the incident to any assumed point in the refracted ray.

When the incident ray is parallel to the axis, $d\rho = - dx$, and this equation becomes

$$md\rho' = dx.$$

(206.) To determine the curve which will refract rays proceeding from a point *accurately* to a point, we have but to consider $(\alpha\beta)$, $(\alpha'\beta')$, as given points in the equation (204.), and integrate on that supposition.

For the sake of simplification let the focus of the refracted rays be taken as the origin of the co-ordinates, or let $\alpha' = 0$, $\beta' = 0$, in the differential equation; and, integrating,

$$\sqrt{(\alpha - x)^2 + (\beta - y)^2} + m\sqrt{x^2 + y^2} = a,$$

a being the arbitrary constant. This equation is that of a curve of the fourth dimension, which has been called, from its inventor, the *Cartesian oval.* It is evidently equivalent to

$$\rho + m\rho' = a;$$

a result which expresses the fundamental property of the curve, and by means of which it may be easily constructed by points.

The preceding equation may be conveniently transformed to one in polar co-ordinates. For this purpose, let $x = r\cos.\omega$, $y = r\sin.\omega$, and substituting, we find

$$(\alpha - r\cos.\omega)^2 + (\beta - r\sin.\omega)^2 = (a - mr)^2.$$

As the axis of abscissæ is perfectly arbitrary, we may take for it the line joining the two foci; in which case $\beta = 0$. Now, if x' denote the co-ordinate of the intersection of the curve with this axis, or the value of x corresponding to $y = 0$,

$$\alpha - x' + mx' = a.$$

When the constant $a = 0$; *i. e.* when $a - x' = -mx'$, or the distance of the focus of incident rays from the surface is to that of reflected rays from the same in the ratio of $m : 1$, the refracting curve becomes the *circle;* for, in this case, the equation is

$$(a - x)^2 + y^2 = m^2(x^2 + y^2).$$

And if we transfer the origin to the point, whose abscissa $= \frac{a}{1-m^2}$, this equation becomes

$$y^2 + x^2 = \frac{m^2a^2}{(m^2-1)^2},$$

the equation of a circle referred to the centre, whose radius $= \frac{ma}{m^2-1}$.

(207.) When the incident rays are parallel, we must return to the differential equation (204.), in which, making $\alpha' = 0$, $\beta' = 0$, and integrating, as before, there is

$$x + m\sqrt{x^2 + y^2} = \text{const.}$$

Now, x' being the value of x, when $y = 0$; const. $= (m + 1)x'$. Wherefore, making this substitution, and transforming the equation to polar co-ordinates, we have

$$r = \frac{(m+1)x'}{m + \cos.\omega}.$$

The equation of a conic section, whose *excentricity* $= \frac{1}{m}$.

It is therefore an *ellipse* when $m > 1$, or the refracting medium *denser* than the surrounding medium; a *hyperbola,* when $m < 1$, or the refracting medium *rarer.*

From the preceding it will readily appear that, by the combination of *spheroidal* with plane or spherical surfaces, it is always possible to construct a lens which shall be *perfectly aplanatic* for parallel rays. For it appears from what has been said, that when parallel rays are incident from the rarer into the denser medium, if the surface bounding the latter be that generated by the revolution of an *ellipse,* whose excentricity $= \frac{1}{m}$, about its major axis, the rays will be refracted accurately to the farther focus. If then the second surface, bounding the medium, be a portion of a spherical surface, whose centre is that focus and radius any line less than the distance of that focus from the first surface, the rays will be incident upon it perpendicularly, and therefore undergo no refraction there; and the *meniscus*, thus formed, will be perfectly aplanatic for parallel rays.

Again, when parallel rays pass from the denser into the rarer medium, if the bounding surface be a *hyperboloid* generated by the revolution of a hyperbola whose excentricity is $\frac{1}{m}$, about its major axis, the rays will be refracted accurately to the farther focus. If, therefore, a *plano-convex* lens be constructed, whose second surface is the hyperboloid just mentioned, and first surface a plane perpendicular to its axis, it will be perfectly aplanatic for parallel rays, the rays undergoing no refraction at its first surface.

(208.) As an application of this theory, let us take the case in which rays diverging from a point are incident on a *plane* refracting surface.

The perpendicular from the radiant point on the plane being taken as the axis of abscissæ, there is $\beta = 0$; and, if we take the point in which this perpendicular meets the plane as the origin, the equation of the plane will be simply

$$x = 0, \quad \text{whence also } dx = 0.$$

And these substitutions being made in the equation of the refracted ray (204.), it is reduced to

$$m\varrho(\beta' - y) - \varrho' y = 0;$$

or, if we transfer $\varrho' y$ to the other side of the equation, square both sides, and substitute for ϱ^2 and ϱ'^2 their values, which are, in this case, $\alpha^2 + y^2$ and $\alpha'^2 + (\beta' - y)^2$, respectively, we have

$$(\beta' - y)\left[m^2\alpha^2 + (m^2 - 1)y^2\right]^{\frac{1}{2}} = \alpha' y,$$

the complete equation of the refracted ray.

To get the intersection of the refracted ray with the axis, let $\beta = 0$ in this equation, and there is

$$\alpha' = \left[m^2\alpha^2 + (m^2 - 1)y^2\right]^{\frac{1}{2}}.$$

For rays indefinitely near the axis in their incidence, $y = 0$; and

$$\alpha' = m\alpha,$$

agreeing with the results already obtained (112.) and (113.).

(209.) Let us now consider the case in which the refracting surface is a *sphere*.

Taking the line joining the centre with the radiant point as the axis of abscissæ, $\beta = 0$; and if the centre be taken as the origin, the equation of the circle is

$$y^2 + x^2 = r^2, \text{ whence } p = \frac{dy}{dx} = -\frac{x}{y}.$$

Making these substitutions in equation (204.), it becomes

$$\varrho'\alpha y + m\varrho(\alpha' y - \beta' x) = 0.$$

From which, by squaring and substituting for ϱ^2 and ϱ'^2 their values, we obtain the relation between the co-ordinates of the refracted ray.

To find the point in which this ray meets the axis, let $\beta' = 0$, and the equation becomes

$$\varrho'\alpha + m\varrho\alpha' = 0,$$

a result already obtained (134.). If we square and substitute for ϱ and ϱ' their values, this becomes

$$\alpha^2\left[(\alpha' - x)^2 + y^2\right] = m^2\alpha'^2\left[(\alpha - x)^2 + y^2\right];$$

an equation which agrees with that from which we have already deduced the whole theory of aberrations, foci, &c. (146.).

For rays indefinitely near the axis, we must make $y = 0$, and therefore $x = r$, in this equation, which is thus reduced to

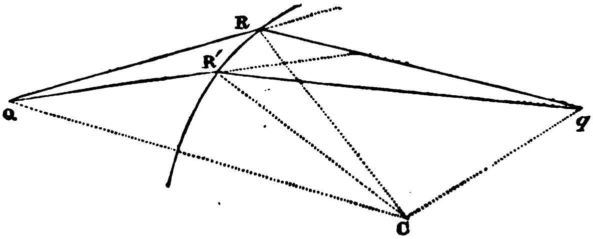
$$m\alpha'(\alpha - r) = \alpha(\alpha' - r);$$

whence we obtain

$$\frac{m}{\alpha} - \frac{1}{\alpha'} = \frac{m-1}{r}.$$

II.

Of Caustics produced by Refraction at any curved Surfaces.



(210.) Let QR, QR′, be two incident rays indefinitely near, Rq, R′q, the refracted rays meeting in q, which is therefore a point of the caustic; RC, R′C, two normals to the curve at these indefinitely near points, meeting in C, which is therefore the centre of the osculating circle. Then, if we denote QC and qC, the distances of the foci of incident and refracted rays from that centre, by D and d; QR and Rq, the distances of the same foci from the point of incidence, by ρ and ρ'; RC, the radius of the osculating circle, by r, and the angles of incidence and refraction by θ and θ'; we have in the triangles QRC, qRC,

$$D^2 = r^2 + \rho^2 + 2r\rho \, . \cos.\theta,$$
$$d^2 = r^2 + \rho'^2 - 2r\rho' . \cos.\theta';$$

and differentiating, as before, considering D, d, and r as constant,

$$(\rho + r . \cos.\theta)d\rho - r\rho \, . \sin.\theta \, . d\theta = 0,$$
$$(\rho' - r . \cos.\theta')d\rho' + r\rho' . \sin.\theta' . d\theta' = 0.$$

Now the angles θ and θ' are connected by the relation

$$\sin.\theta = m . \sin.\theta'; \text{ whence } \cos.\theta \, . d\theta = m . \cos.\theta' . d\theta'.$$

Again, the differentials $d\rho$ and $d\rho'$ are related; for

$$\sin.\theta = \frac{d\rho}{ds}, \quad \sin.\theta' = -\frac{d\rho'}{ds},$$

ds being the differential of the arc of the refracting curve; and substituting these values in the equation $\sin.\theta = m.\sin.\theta'$, there is

$$d\rho + m . d\rho' = 0.$$

Accordingly, if we divide the former of the differential equations obtained above by the latter, and substitute in the result for $\frac{d\rho}{d\rho'}$ and $\frac{d\theta}{d\theta'}$ their values $-m$, and $m . \frac{\cos.\theta'}{\cos.\theta}$, derived from the equations just obtained, we find

$$\frac{\rho + r . \cos.\theta}{\rho' - r . \cos.\theta'} = \frac{\rho}{\rho'} . \frac{\sin.\theta}{\sin.\theta'} . \frac{\cos.\theta'}{\cos.\theta}.$$

From this equation we obtain the value of ρ'; for, if we expand it by taking away the denominators, we have

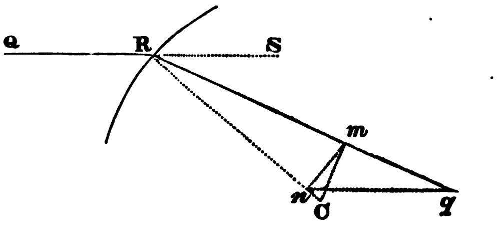
$$r[\rho . \sin.\theta . \cos.^2\theta' + \rho' . \sin.\theta' . \cos.^2\theta] = \rho\rho' . \sin.(\theta - \theta');$$

from which, dividing by $r\rho\rho'$, we obtain

$$\frac{\sin.\theta' . \cos.^2\theta}{\rho} + \frac{\sin.\theta . \cos.^2\theta'}{\rho'} = \frac{\sin.(\theta - \theta')}{r}.$$

(211.) When the incident rays are parallel, $\frac{1}{\rho} = 0$, and this equation gives for the value of ρ',

$$\rho' = r . \frac{\sin.\theta . \cos.^2\theta'}{\sin.(\theta - \theta')}.$$



This value is readily constructed. From the centre of the osculating circle C, let the perpendicular Cm be drawn to the refracted ray, from m the perpendicular mn on the radius of curvature, and from n the line nq, parallel to the incident ray QR; then q, the point of intersection of this line with the refracted ray, is the point of the caustic;

$$\text{for } \mathrm{R}n = \mathrm{R}m \,.\, \cos.\theta' = r \,.\, \cos.^2\theta',$$

$$\text{and } \mathrm{R}q = \mathrm{R}n \,.\, \frac{\sin.\mathrm{R}nq}{\sin.\mathrm{R}qn} = \mathrm{R}n \,.\, \frac{\sin.\mathrm{SR}n}{\sin.\mathrm{SR}q} = \frac{r.\cos.^2\theta' \,.\, \sin.\theta}{\sin.(\theta-\theta')}.$$

(212.) If we expand the second member of equation (210.), divide by $\sin.\theta'$, and put for $\frac{\sin.\theta}{\sin.\theta'}$ its value m, we find

$$\frac{\cos.^2\theta}{\rho} + m.\frac{\cos.^2\theta'}{\rho'} = \frac{m \,.\, \cos.\theta' - \cos.\theta}{r},$$

a form, perhaps, more convenient than the former.

When the refracting surface is *plane*, r is infinite; the second member of this equation therefore vanishes, and it is reduced to

$$\rho' \,.\, \cos.^2\theta + m\rho \,.\, \cos.^2\theta' = 0.$$

Again, let the refracting curve be the *logarithmic spiral*, and the radiant point its pole.

As the angle contained by the radius-vector with the tangent or normal, in this curve, is constant, it follows that θ, and therefore also θ', is constant. Again, the radius of curvature in this curve is

$$r = \frac{\rho}{\cos.\theta}.$$

And if we substitute this value in the equation preceding, we find

$$\frac{\rho}{\rho'} = \frac{\cos.\theta}{\cos.\theta'}\left(1 - \frac{2}{m}\,\frac{\cos.\theta}{\cos.\theta'}\right).$$

From which it appears that $\frac{\rho}{\rho'}$ is constant, since θ and θ' are so. Wherefore in the triangle formed by the lines joining the point of incidence in the refracting curve and the foci of incident and refracted rays, the ratio of two of the sides, ρ and ρ', is given, as is also the contained angle, $\theta - \theta'$; hence the triangle is given in species, and, consequently, the angle contained by the refracted ray with the line joining the foci of incident and refracted rays is determined. But this is the angle contained by the radius-vector of the caustic with its tangent; the caustic curve, therefore, is also a logarithmic spiral, having the radiant point as its pole.

(213.) We shall now derive the theory of *diacaustics* from the consideration of the equation of the refracted ray.

When the incident rays are *parallel*, making $dy = pdx$ in the equation of the refracted ray (204.), it is

$$\rho' + m\left[(\alpha' - x) + p(\beta' - y)\right] = 0,$$

which is equivalent to

$$m \cdot \frac{d\rho'}{dx} = 1.$$

Now, differentiating this equation relatively to x and y, as in the case of reflexion, there is

$$\frac{d\rho'}{dx} + m\left[(\beta' - y)q - (1 + p^2)\right] = 0.$$

And multiplying by m, and putting 1 for $m\,\frac{d\rho'}{dx}$,

$$m^2(\beta' - y)q = m^2(1 + p^2) - 1.$$

And if we eliminate x and y between this equation and that of the refracted ray above given, combined with the equation of the refracting curve, the resulting equation, containing α' and β' only, will be the equation of the caustic.

In the case of *reflexion*, $m = -1$, and these equations become

$$\rho' = (\alpha' - x) + p(\beta' - y), \quad (\beta' - y)q = p^2,$$

as we have already obtained (89.).

If the refractive power be *infinite*, or m infinite, it is evident that the refracted ray must coincide with the normal, and therefore the caustic with the *evolute*. In fact, the equations become in this case

$$\alpha' - x + p(\beta' - y) = 0, \quad (\beta' - y)q = 1 + p^2;$$

$$\text{or, } \beta' - y = \frac{1+p^2}{q}, \quad \alpha' - x = -p \,.\, \frac{1+p^2}{q};$$

the well known expressions of the co-ordinates of the evolute.

(214.) As an application of the preceding equations, let it be required to find the caustic where parallel rays are incident upon a spherical refracting surface.

The equation of the refracting curve, in this case, is

$$x^2 + y^2 = r^2,$$

whence we obtain

$$p = -\frac{x}{y}, \quad 1 + p^2 = \frac{r^2}{y^2}, \quad q = -\frac{r^2}{y^3};$$

and these values being substituted in the equations of the preceding article, they become

$$\varrho' y + m(\alpha' y - \beta' x) = 0,$$

$$m^2 r^2 \beta' = y^3.$$

The equation resulting from the elimination of x and y from these equations, combined with that of the circle itself, will be that of the caustic required.

(215.) When the incident rays diverge from a point, if that point be taken as the origin of the co-ordinates by making $\alpha = 0$, $\beta = 0$; the equation of the refracted ray is

$$m\varrho\left[(\alpha - x) + p(\beta - y)\right] - \varrho'(x + py) = 0,$$

omitting the traits over α', β', as no longer required. And if we differentiate this equation relatively to x and y, the co-ordinates of the refracting curve; and observe that

$$(\alpha - x) + p(\beta - y) = -\varrho' \,.\, \frac{d\varrho'}{dx}, \quad x + py = \varrho \,.\, \frac{d\varrho}{dx},$$

we obtain

$$m\varrho' \cdot \frac{d\varrho}{dx} \cdot \frac{d\varrho'}{dx} + m\varrho\left[1 + p^2 - (\beta - y)q\right]$$

$$+ \varrho \cdot \frac{d\varrho}{dx} \cdot \frac{d\varrho'}{dx} + \varrho'\left[1 + p^2 + yq\right] = 0,$$

or collecting analogous terms,

$$\left[m\varrho(\beta - y) - \varrho' y\right] q = (m\varrho + \varrho')\,(1 + p^2) + (\varrho + m\varrho')\frac{d\varrho}{dx} \cdot \frac{d\varrho'}{dx}.$$

Now the equation of the refracted ray may be put under the form

$$\varrho' = m\varrho\left[\frac{\alpha + p\beta}{x + py} - 1\right],$$

from which we have

$$m\varrho + \varrho' = m\varrho\frac{\alpha + p\beta}{x + py}, \quad \varrho + m\varrho' = m^2\varrho \cdot \frac{\alpha + p\beta}{x + py} + (1 - m^2)\varrho.$$

Again, in virtue of the same equation, there is

$$\frac{d\varrho'}{dx} = -\frac{1}{m} \cdot \frac{d\rho}{dx},$$

$$\therefore \frac{d\varrho}{dx} \cdot \frac{d\varrho'}{dx} = -\frac{1}{m} \cdot \left(\frac{d}{dx}\right)^2 = -\frac{1}{m} \cdot \frac{(x + py)^2}{x^2 + y^2}.$$

And if we make these substitutions in the differential equation, multiply the result by $m(x + py)\,(x^2 + y^2)$, and observe that

$$(1 + p^2)\,(x^2 + y^2) - (x + py)^2 = (px - y)^2;$$

we shall have finally

$$m^2(x^2 + y^2)\,(\beta x - \alpha y)\,q$$

$$= m^2(px - y)^2\,(\alpha + p\beta) + (m^2 - 1)\,(x + py)^3.$$

In the case of reflexion, $m^2 - 1 = 0$, and this equation becomes

$$(x^2 + y^2)\,(\beta x - \alpha y)\,q = (\alpha + p\beta)\,(px - y)^2,$$

as we have already obtained (97.)

(216.) The caustic curve being the locus of the intersections of the successive refracted rays, each ray, it is evident, must be a tangent to the curve at the point in which it meets it. But the tangent of the angle which the tangent to the caustic makes with the axis of abscissæ is $\frac{d\beta'}{d\,'}$; wherefore ω' being the angle which the refracted ray makes with the same, we have, as in the case of reflexion, $\frac{d\beta'}{d\alpha'} = \tan.\omega'$, or,

$$d\beta' \cdot \cos.\omega' - d\alpha' \cdot \sin.\omega' = 0.$$

Again, if we differentiate the equations

$$(\beta - y)^2 + (\alpha - x)^2 = \varrho^2,\ (\beta' - y)^2 + (\alpha' - x)^2 = \varrho'^2;$$

and substitute in the result, for $\alpha - x$, $\beta - y$, $\alpha' - x$, $\beta' - y$, their values, $\varrho \cos.\omega$, $\varrho \sin.\omega$, $\varrho' \cos.\omega'$, $\varrho' \sin.\omega'$; there is,

$$- dy \,.\, \sin.\omega - dx \,.\, \cos.\omega = d\varrho,$$

$$(d\beta' - dy) \sin.\omega' + (d\alpha' - dx) \cos.\omega' = d\varrho'.$$

Now if we multiply the latter of these equations by m, and then add, and observe that

$$dx \,.\, \cos.\omega + dy \,.\, \sin.\omega + m\,(dx \,.\, \cos.\omega' + dy \,.\, \sin.\omega') = 0,$$

we shall have

$$d\beta' . \sin.\omega' + d\alpha' . \cos.\omega' = \frac{d\varrho}{m} + d\varrho'.$$

Finally, if we square and add together this equation and that obtained above, we have

$$d\beta'^2 + d\alpha'^2 = \left(\frac{d\varrho}{m} + d\varrho'\right)^2 = dz^2,$$

dz being the differential of the arc of the caustic curve: hence

$$dz = \frac{d\varrho}{m} + d\varrho'$$

$$\therefore z = \frac{\varrho}{m} + \varrho' + \text{const.}$$

The caustic curve, therefore, is always *rectifiable*, if the refracting curve be an *algebraic* curve.

(217.) As an exemplification of this theory, let us seek the caustic produced by refraction at a *plane* surface.

The equation of the refracted ray, in this case, we have already found, is

$$(\beta' - y)\left[m^2\alpha^2 + (m^2-1)y^2\right]^{\frac{1}{2}} = \alpha' y;$$

in which α denotes the distance of the radiant point from the plane. If we divide this equation by y, and differentiate, relatively to y only, we find,

$$m^2\alpha^2\beta' + (m^2-1)\,y^3 = 0.$$

Now, if we multiply the former of these equations by $m^2\alpha^2$, and substitute for $m^2\alpha^2\,(\beta'-y)$ its value, $-y\left[m^2\alpha^2 + (m^2-1)y^2\right]$, derived from the latter, it becomes

$$m^2\alpha^2\alpha' = -\left[m^2\alpha^2 + (m^2-1)\,y^2\right]^{\frac{3}{2}}.$$

Finally, if we raise both sides of this equation to the power $\frac{2}{3}$, substitute for y its value in β', derived from the differential equation, and divide the result by $(m\alpha)^{\frac{4}{3}}$, it becomes

$$\alpha'^{\frac{2}{3}} + (\sqrt{1-m^2}\,.\,\beta')^{\frac{2}{3}} = (m\alpha)^{\frac{2}{3}};$$

which is the equation of the *evolute* of a *conic section*, having its centre at the origin, and focus at the radiant point. The conic section is an *ellipse* or *hyperbola*, according as m is less or greater than unity.

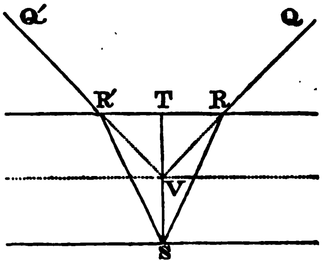
CHAPTER VIII.

OF REFLEXION AND REFRACTION COMBINED.

(218.) It has been already observed that, when a beam of light is incident upon the surface of any transparent medium, a portion of the incident light is, in all cases, reflected at the bounding surface, and never enters the medium. If, again, the portion which enters the medium meet with a second bounding surface, part of it will be reflected there; and thus, meeting the first surface a second time, it will again be subdivided into two portions, one of which will re-emerge into the original medium, and the other suffer a second reflexion.

As this combination of reflexion with refraction is the cause of some remarkable phenomena, and occurs also, more distinctly, in the artificial combinations employed for optical purposes, it becomes necessary to examine its laws.

(219.) When a ray of light is incident upon a medium bounded by two parallel plane surfaces, let it be required to determine the direction of the emergent ray, after being twice transmitted through the first surface, and once reflected at the second.



Let QRSR'Q' be the course of the ray, entering the first surface of the medium at R, reflected by the second at S, and again transmitted through the first at R'; and let ST be the perpendicular to the surface at the point of reflexion. Then, since the angles at S are equal, the angles SRT, SR'T, which the portions of the ray within the medium make with the first surface, are equal; consequently, the angles which the incident and emergent rays QR and Q'R' form with that surface are also equal; and therefore these rays produced must meet the perpendicular ST in the same point V, and contain with it equal angles. Hence the direction of the emergent ray is the same as if it had undergone only a single reflexion, at a plane drawn parallel to the two surfaces at a distance from the first equal to TV.

To determine the distance of this imaginary plane, we have

$$TV = ST \cdot \frac{\text{cotan.}TVR}{\text{cotan.}TSR} = ST \cdot \frac{\tan.TSR}{\tan.TVR}.$$

Now the angles TVR, TSR, are the angles of incidence and refraction into the medium; wherefore, denoting them by θ and θ' respectively, and the interval between the two surfaces by δ, the distance of the imaginary plane from the first surface is

$$\delta \cdot \frac{\tan.\theta'}{\tan.\theta}.$$

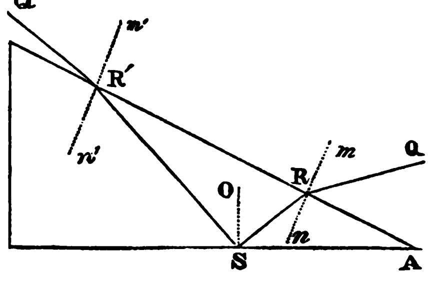
Hence it is obvious that the distance of this plane varies with the incidence of the ray. When the ray is incident perpendicularly, the ultimate value of the ratio $\frac{\tan.\theta}{\tan.\theta'}$ is $\frac{\sin.\theta}{\sin.\theta'}$, or μ; and the distance of the reflecting plane is

$$\frac{\delta}{\mu}.$$

It is evident that the ray can never undergo a total reflexion at the second surface; for the angle of incidence on the second surface is equal to the angle of refraction at the first, and therefore less than the limiting angle of total reflexion.

(220.) Let it be required to determine the course of a ray of light which, entering the first surface of a prism, is reflected at the second, and again transmitted through the first.

Let QRSR'Q' be the course of such a ray, mn and $m'n'$ the perpendiculars to the first surface at the points of transmission, and so the perpendicular to the second at the point of reflexion. Then if the angles QRm, SRn,



be denoted by φ and ψ, the angles Q'R'm', SR'n', by φ' and ψ', there is

$$\sin.\varphi = \mu.\sin.\psi, \qquad \sin.\varphi' = \mu.\sin.\psi'.$$

Again, if the angle of incidence on the second surface, RSO, be denoted by θ, and the angle of the prism by ϵ,

$$\psi = \theta - \epsilon, \qquad \psi' = \theta + \epsilon,$$

and subtracting

$$\psi' - \psi = 2\epsilon;$$

which equation, combined with the two already obtained, will determine the direction of the emergent ray when that of the incident ray is known.

From these equations it appears that the inclination of the emergent ray to the surface is the same as in the case of transmission through a prism whose refracting angle is 2ϵ, double the refracting angle of the given prism.

The total deviation of the ray is equal to the sum of the angles which the incident and emergent rays contain with the first surface of the prism; that is, denoting the deviation by δ,

$$\delta = \pi - (\varphi + \varphi').$$

If the values of ψ and ψ' be added together, there is

$$\psi + \psi' = 2\theta.$$

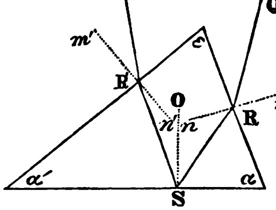
Hence, if one of the angles ψ or ψ' be less than θ, the other will be greater; and, consequently, if θ be equal to or greater than the angle of total reflexion, the ray cannot be transmitted at the corresponding point. It follows, therefore, as in the last case, that a ray of light cannot be twice transmitted through the first surface, and suffer total reflexion at the second. The case is different if the ray emerge through a third side of the prism; as will appear from what follows.

(221.) Let it be required to determine the direction of a ray of light which, entering the side of a prism, is reflected at the base and emerges through the remaining side.

Let QRSR'Q' be the course of such a ray; mn, $m'n'$, the perpendiculars to the surfaces at the points of incidence and emergence; and so the perpendicular to the base at the point of reflexion. Then, if the angles QRm, SRn, be denoted by

φ and ψ, the angles Q'R'm', SR'n', by φ' and ψ', as before, there is

$$\sin.\varphi = \mu . \sin.\psi, \quad \sin.\varphi' = \mu . \sin.\psi'.$$



Again, if the angle RSO = R'SO be denoted by θ, and the angles contained by the two sides of the prism with the base by α and α' respectively, we have

$$\psi + \theta = \alpha, \quad \psi' + \theta = \alpha',$$

and subtracting

$$\psi' - \psi = \alpha' - \alpha;$$

which equation, combined with the two already obtained, will determine the direction of the emergent ray when that of the incident ray is known.

It appears from these equations that the inclination of the emergent ray to the surface will be the same as if the ray had been transmitted through a prism whose refracting angle is $\alpha' - \alpha$, the difference of the angles at the base of the given prism.

If the prism be isosceles, or $\alpha' = \alpha$, there will be $\psi' = \psi$, and therefore $\varphi' = \varphi$. Hence, when a ray is transmitted through the sides of an isosceles prism, and suffers an intermediate reflexion at its base, the ray will emerge inclined to the surface at the same angle as at its incidence.

The sum of the deviations of the ray at incidence and emergence is $\varphi + \varphi' - (\psi + \psi')$; and the deviation at the point of reflexion is $\pi - 2\theta$. Wherefore, adding, and observing that $\psi + \psi' + 2\theta = \alpha + \alpha'$, the total deviation is

$$\delta = \varphi + \varphi' + \pi - (\alpha + \alpha') = \varphi + \varphi' + \varepsilon,$$

ε denoting the vertical angle of the prism.

(222.) From the equations

$$\psi = \alpha - \theta, \quad \psi' = \alpha' - \theta,$$

it is obvious that it will be always possible to take the angle θ of such value, that the ray shall be transmitted at the two sides, and suffer total reflexion at the base. For, that this should take place, it is only necessary that θ should be equal to or greater than $\theta_{\prime}$, the angle of total reflexion, and ψ and ψ' each less than the same.

In order that the ray should be incident and emergent perpendicularly, it is necessary that ψ and ψ' should be each equal to nothing, or that

$$\alpha = \theta = \alpha'.$$

And if, moreover, the ray undergoes a total reflexion at the base, θ must be equal to or greater than $\theta_{\prime}$. Hence, if the ray be incident perpendicularly on the side of an isosceles prism, of which the base angles are equal to or greater than the angle of total reflexion, it will suffer total reflexion at the base, and emerge perpendicularly; and accordingly the course of the ray will be the same as if it had been reflected by a plane surface, whose position is that of the base of the prism.

When the ray just suffers total reflexion,

$$\psi = \alpha - \theta_{\prime}, \qquad \psi' = \alpha' - \theta_{\prime}.$$

Hence, if either of the angles at the base of the prism, α or α', be equal to $\theta_{\prime}$, the angle of total reflexion, ψ or ψ' will be nothing, and the ray which just suffers total reflexion, incident or emergent perpendicularly. When α or α' is greater than $\theta_{\prime}$, ψ or ψ', and therefore ϕ or ϕ', will be positive, and the incident or emergent ray inclined from the perpendicular towards the vertex; and, on the contrary, when α or α' is less than $\theta_{\prime}$, the incident or emergent ray, which just suffers total reflexion at the base, will be inclined towards the base. The value of the angle of incidence or emergence in this case is readily determined: for then is

$$\sin.\phi = \mu . \sin.\psi = \mu . \sin.(\alpha - \theta_{\prime}).$$

Wherefore, expanding the second term, and substituting for $\sin.\theta_{\prime}$, $\cos.\theta_{\prime}$, their values $\frac{1}{\mu}$, and $\sqrt{1 - \frac{1}{\mu^2}}$, we have

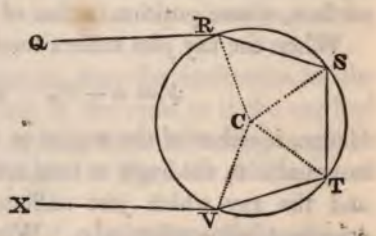
$$\sin.\phi = \sqrt{\mu^2 - 1} . \sin.\alpha - \cos.\alpha.$$

The sine of the angle of emergence ϕ' will be a similar function of α'.

(223.) Let it be required to determine the course of a ray of light which enters a sphere and emerges after any number of internal reflexions.

It is evident that the ray will continue in the same plane throughout, namely, the plane containing the incident ray and the centre of the sphere. For the refracted ray, both at incidence and emergence, is in the plane containing the incident ray and the radius at the point of incidence; and each reflected ray is in the plane containing the incident ray and radius drawn to point of incidence. Hence we may confine our attention to the great circle containing the incident ray.

Let RSTV be this circle, and QRSTVX the course of the ray which is incident and emergent at R and V, and reflected at S, T, &c. Then it is evident that the angles contained by each chord with the radii drawn to its extremities, CRS and CSR, CST and CTS, CTV and CVT, are respectively equal; as are also the angles CSR and CST, CTS and CTV, &c. contained by each radius with the adjacent chords, these being the angles of incidence and reflexion. Hence all the angles contained by the direction of the ray within the sphere with radius are equal; wherefore the angles CRS and CVT at incidence and emergence are equal, and accordingly the ray emerges under an angle equal to that of incidence.



It is obvious that the portions of the ray within the sphere, RS, ST, TV, &c., and therefore the arches which they subtend, are all equal; the triangles CRS, CST, CTV, &c. being equal in every respect.

It appears also from what has been said that the ray can never undergo total reflexion within the sphere, the angle of incidence at each point of reflexion being equal to the angle of refraction into the sphere, and therefore less than the limiting angle of total reflexion.

If θ and θ' denote the angles of incidence and refraction into the sphere, the angle of deviation at the point of incidence is $\theta - \theta'$; and, as the deviation is the same at emergence, the sum of the deviations at incidence and emergence is $2(\theta - \theta')$. Again, the deviation at each point of reflexion is equal to $\pi - 2\theta'$, each angle of reflexion being equal to the angle of refraction into the sphere. Wherefore, if the number of reflexions which the ray undergoes within the sphere be denoted by n, the sum of the deviations at the points of reflexion will be $n\pi - 2n\theta'$. Therefore, adding this to the sum of the deviations at incidence and emergence, the total deviation is

$$\delta = n\pi + 2\theta - 2(n + 1)\theta'.$$

(224.) To find when the deviation of the ray is a maximum or minimum, let its differential be taken equal to nothing, and there is

$$d\theta = (n + 1)d\theta'.$$

But we have also

$$\sin.\theta = \mu . \sin.\theta', \qquad \cos.\theta . d\theta = \mu . \cos\theta' . d\theta';$$

and, substituting in the latter the value of $d\theta$ just obtained,

$$(n + 1)\cos.\theta = \mu . \cos.\theta';$$

which equation, combined with the former, will determine the angle of incidence when the deviation is a maximum or minimum. To effect the elimination required, we have only to square these equations and add them together, and we find

$$(n + 1)^2\cos.^2\theta + \sin.^2\theta = \mu^2;$$

$$\text{or, } \left[(n + 1)^2 - 1\right]\cos.^2\theta = \mu^2 - 1;$$

$$\therefore \cos.\theta = \sqrt{\frac{\mu^2 - 1}{n(n+2)}}.$$

(225.) A pencil of rays being incident nearly perpendicularly upon a medium bounded by parallel plane surfaces, let it be required to determine the focus of the emergent pencil after reflexion at the second surface and a double transmission through the first.

Let d denote the distance of the focus of the incident pencil from the first surface; d', d'', and d''', the distances of the foci of the pencil after refraction by the first surface, reflexion at the second, and a second refraction by the first; and let δ denote the interval between the surfaces. Then for the refraction at the first surface, we have the relations,

$$d' = \mu d, \quad d'' = \mu d'''.$$

Again, the distances of the foci of the incident and reflected rays from the second surface are $d' + \delta$ and $d'' + \delta$ respectively; and since their sum is equal to nothing, the equation of the reflexion at the second surface is

$$d' + d'' + 2\delta = 0.$$

If then we substitute in this for d' and d'' their values obtained from the two preceding, and divide the result by μ, there is

$$d + d''' + \frac{2\delta}{\mu} = 0,$$

an equation expressing the relation between the distances of the foci of incident and emergent rays.

From this equation it appears that the effect produced is the same as if the incident pencil had suffered reflexion only at a plane, whose distance from the first surface is $\frac{\delta}{\mu}$; a conclusion which might have been deduced immediately from (219.).

As the greater part of the light, under ordinary circumstances, will be transmitted at the second surface, the effect produced by this reflexion at the second surface combined with two transmissions through the first will not be distinctly observed, unless the reflexion of the second surface be strengthened by coating it with mercurial amalgam, as in the common looking-glass.

(226.) The whole light, however, which is thus thrown back to the first surface, will not emerge there. A portion of it will be reflected there, again reflected at the second surface, and, finally, emergent at the first, having undergone three reflexions. Part, again, will be thrown back and emerge after five reflexions, &c. And thus there will be an indefinite series of foci of pencils emergent after one, three, five, &c. reflexions; the condensation of the light in these foci, and therefore in the images they form, decreasing rapidly. These phenomena may be easily observed by looking obliquely at the image of a candle formed by a glass mirror.

The positions of these foci are easily ascertained. Thus, when the pencil undergoes two reflexions at the second surface, and one at the first, let d denote, as before, the distance of the focus of incident rays from the first surface; $d^{\prime}$, $d^{\prime\prime}$, $d^{\prime\prime\prime}$, d^{iv}, and d^{v}, the distances of the successive foci of the pencil after the several modifications which it undergoes, these distances being measured from the first surface towards the incident light: then it will be easily seen that these distances are connected by the following equations:

$$\begin{gathered} d^{\prime} = \mu d \\ d^{\prime} + d^{\prime\prime} + 2\delta = 0 \\ d^{\prime\prime} + d^{\prime\prime\prime} = 0 \\ d^{\prime\prime\prime} + d^{\text{iv}} + 2\delta = 0 \\ d^{\text{iv}} = \mu d^{\text{v}}. \end{gathered}$$

To eliminate $d^{\prime}$, $d^{\prime\prime}$, $d^{\prime\prime\prime}$, and d^{iv}, from these equations, we have only to change the signs of the second and fourth, and then add them together; and dividing the result by μ, we obtain

$$d + d^{\text{v}} + \frac{4\delta}{\mu} = 0;$$

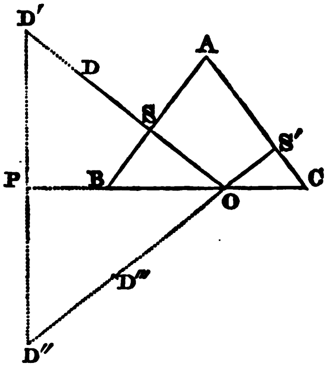
from which it appears that the effect is the same as if the rays had undergone a single reflexion at the surface whose distance from the first is $\frac{2\delta}{\mu}$.

In like manner, when the pencil emerges after three reflexions at the second surface, and two at the first, we should find that the position of the focus of the emergent pencil is the same as if the incident pencil had been simply reflected at the surface whose distance from the first is $\frac{3\delta}{\mu}$.

It appears then that there will be a series of foci of emergent rays, suffering two transmissions and one reflexion, two transmissions and three reflexions, two transmissions and five reflexions, &c., whose virtual reflecting planes are, in succession, distant from one another and from the first surface of the medium by the interval $\frac{\delta}{\mu}$. Hence it follows that these foci, together with the focus of rays reflected at the first surface only, are ranged along the same right line at equal intervals, their distances from each other being equal to $\frac{2\delta}{\mu}$.

In plate glass $\mu = \frac{3}{2}$ nearly; wherefore the interval between the successive foci formed by a plate-glass speculum is four-thirds of the thickness of the plate.

(227.) A small pencil of rays being incident perpendicularly upon the side of an isosceles prism, and emergent at the other side after reflexion at the base, it is required to find the focus of the emergent pencil.



Let D be the focus, and DS the axis of the pencil incident upon the first surface of the prism BAC; then if we take on that line $D'S = \mu . DS$, D' will be the focus of the pencil after refraction by the first side, and therefore the focus of the pencil incident upon the base at O. If, then, we let fall from this point the perpendicular D'P upon the base, and produce it equally to D'', the pencil will diverge from the focus D'', after reflexion by the base, and be incident perpendicularly upon the remaining side at S' (221.). Accordingly, the focus of the emergent pencil is obtained by taking, in the line D''OS', a point D''' such that $D''S' = \mu . D'''S'$.

Now, on account of the equality of the lines D'O and D''O,

$$D''S' = D'O + OS' = D'S + OS + OS';$$

$$\text{or,} \quad \mu . D'''S' = \mu . DS + OS + OS'.$$

But, in an isosceles triangle, the sum of the perpendiculars let fall from any point of the base upon the sides is constant, and equal to the perpendicular let fall from either extremity of the base upon the opposite side. If then this perpendicular be denoted by p, we have $\text{os} + \text{os}' = p$. Wherefore, if DS and D'''S', the distances of the foci of incident and emergent rays from the first and second surfaces respectively, be denoted by δ and δ', we have finally

$$\delta' = \delta + \frac{p}{\mu}.$$

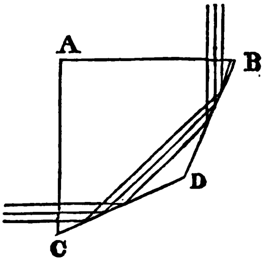
Hence the difference between the distances of the foci of incident and emergent rays from the two sides of the prism is equal to the thickness of the medium traversed by the pencil divided by its index of refraction. The analogy between this result and that of (225.) will be at once observed.

When the dimensions of the prism are small in comparison with the distance of the radiant, $\frac{p}{\mu}$ may be neglected in comparison with δ, and the distance of the focus of the emergent pencil from the second surface is *quam proximè* equal to the distance of the radiant point from the first.

The deviation of the axis of the pencil is equal to the vertical angle of the prism.

(228.) From the preceding it is evident that if the pencils diverging from the several points of an object be received perpendicularly upon the side of a right-angled isosceles prism, they will emerge perpendicularly through the other side after undergoing a total reflexion at the base; and the object, whose direction is horizontal with respect to the spectator, will be seen by looking directly downwards. In this manner the image of an object may be made to coincide with a paper placed on a table underneath the prism, and may be traced there with a pencil.

Such is the principle of the *camera lucida* invented by Dr. Wollaston, an instrument of great use in assisting the draftsman. In the prism just described, however, the image will be seen in an inverted position by a spectator having his face towards the object. To obviate this objection, Dr. Wollaston substituted a quadrilateral prism, such as that represented in the adjoining figure, for the triangular one already mentioned. In this prism the angle at A is right; the opposite angle D is a right angle and a half, or 135°; and the angles at B and C are each three-fourths of a right angle, or 67°. 30′. It is evident, then, that the rays which are incident perpendicularly upon the side AC, and reflected successively at CD and DB, will emerge perpendicularly at the remaining side AB; and that the image will be erect, the rays proceeding from the upper part of the object going to the upper part of the image, and *v. v.*



The prism here described is mounted in a brass frame, and attached by its axis to the end of a brass stem, the lower extremity of which may be clamped to a table by means of a screw. The upper surface of the prism, AB, is furnished with an eye-stop having a small aperture, which is to be adjusted so that the aperture shall be as nearly as possible divided equally by the edge of the prism at B: by this means only a small portion of the surface AB, near the edge, is employed, all the rest being covered. This being done, and the side of the prism, AC, placed in a vertical plane and turned towards the object, the observer looks downwards through the aperture in the eye-stop, and sees, at the same time, the image of the object through the uncovered portion of the prism, and the paper on which it is thrown through the remaining part of the aperture.

Since the dimensions of the prism are very small in comparison with the distance of the object, it appears from the preceding article that the distances of the object and its image are very nearly equal. Hence, if the distance of the object from the prism be different from that of the latter from the table, the image of the object will not be thrown on the paper, and the simultaneous vision of the image and paper will be imperfect. To remedy this, the prism is furnished with a convex and a concave lens, the focal length of each being equal to the greatest distance of the prism from the table. When the former of these lenses is used, it is turned up horizontally under the prism, and, the paper being in its principal focus, its image is thrown to an infinite distance, and therefore made to coincide with the image of a remote object formed by the prism. When the concave lens is used, it is placed vertically in front of the first surface of the prism; and the rays proceeding from a distant object, after refraction by the lens, will diverge from an image whose distance is equal to the focal length of the lens: this image will therefore coincide with the paper after refraction by the prism. The convex lens is to be used by long-sighted persons, or those whose eyes require parallel rays; the concave by short-sighted persons, or those whose eyes are adapted to diverging rays.

Such is the adjustment for distant objects: for near objects the adjustment of the distances is completed by varying the distance of the prism from the paper; and to effect this, the stem of the instrument is furnished with a sliding tube by means of which it can be lengthened or shortened at pleasure.

(229.) Let it be required to determine the focus of a pencil of rays which is transmitted through the first surface of a lens, and emerges at the same after suffering reflexion at the second surface.

Let α, β, β' and α', denote the reciprocals of the distances of the foci from the surface, and ϱ and ϱ' the curvatures of the two surfaces. Then, in consequence of the transmission through the first surface, we have the equations

$$\mu(\beta - \varrho) = \alpha - \varrho, \quad \mu(\beta' - \varrho) = \alpha' - \varrho.$$

Again, if the thickness of the lens be neglected as inconsiderable, there is, in virtue of the reflexion at the second surface,

$$\beta + \beta' = 2\varrho'.$$

To eliminate β and β' from these equations we have only to multiply the latter by μ, and substitute in it, for $\mu\beta$, $\mu\beta'$, their values derived from the former; by which means we obtain

$$\alpha + \alpha' = 2\left[\mu\varrho' - (\mu - 1)\varrho\right];$$

an equation expressing the relation between the distances of the foci of the incident and emergent pencils, and from which it appears that the effect is the same as that produced by reflexion only at a surface whose curvature is $\mu\varrho' - (\mu - 1)\varrho$.

Hence this virtual reflecting surface will be concave or convex, according as ϱ' is greater or less than $\frac{\mu - 1}{\mu}\varrho$. When $\varrho' = \frac{\mu - 1}{\mu}\varrho$, $\alpha + \alpha' = 0$; and the effect is the same as that of reflexion at a plane surface.

(230.) If φ denote the value of α', when $\alpha = 0$, or the incident rays parallel, there is

$$\varphi = 2\left[\mu\varrho' - (\mu - 1)\varrho\right];$$

an equation which may be put under the form

$$\varphi = 2\varrho' - 2(\mu - 1)(\varrho - \varrho').$$

From which we learn that the power of this reflecting lens is equal to the excess of the power of the second reflecting surface above double the power of the lens, when used simply to transmit the pencil; a result which is quite in conformity with our preconceived notions, inasmuch as the pencil undergoes a reflexion at the second surface and two transmissions through the lens with an opposite effect.

When the first surface of the lens is *plane*, or $\varrho = 0$,

$$\varphi = 2\mu\varrho';$$

and the effect is the same as that produced by a single reflecting surface whose curvature is $\mu\varrho'$.

When $\varrho' = 0$, or the second surface plane,

$$\phi = - 2(\mu - 1)\varrho;$$

and the curvature of the virtual reflecting surface is $-(\mu - 1)\varrho$.

When the curvatures of the two surfaces are equal and in the same direction, $\varrho' = \varrho$, and

$$\varphi = 2\varrho;$$

and the effect therefore is that of a simple reflexion at either surface, the lens producing in this case no effect whatever.

When the curvatures are equal and opposite, $\varrho' = - \varrho$, and

$$\varphi = - 2(2\mu - 1)\varrho;$$

and the curvature of the equivalent reflecting surface is $-(2\mu - 1)\varrho$.

PART II.

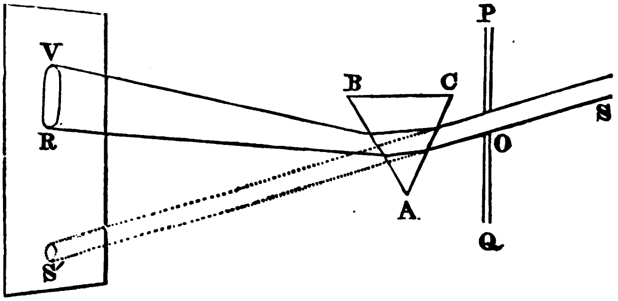
OF COMPOUND OR HETEROGENEOUS LIGHT.

CHAPTER I.

OF THE ANALYSIS OF SOLAR LIGHT.

(231.) We have hitherto treated of *simple* or *homogeneous* light, and considered the modifications which it undergoes in reflexion and refraction. The light of the sun, however, is not of this nature, but is found to be *compound;* each ray of solar light consisting of an infinite number of rays of simple light, differing from each other in *colour* and *refrangibility*. This important discovery we owe to Newton. We shall here briefly describe a few of the principal experiments by which it is established.

If a cylindrical beam of solar light, so, be admitted into a darkened room through a small circular aperture, o, and received on a screen at a distance from the hole, a circular image of the sun, s′, will be there depicted, whose diameter will correspond to that of the hole. If now the beam be received on a triangular glass prism, BAC, placed near the hole, and having its axis perpendicular to the direction of the incident beam, the image of the sun will be thrown upwards by the refraction of the prism into the position RV, and will be no longer white and circular, but coloured and oblong; the sides which are perpendicular to the axis of the prism being rectilinear and parallel, and the extremities, at R and V, semicircular. The breadth of this image, or *spectrum*, as it is usually called, is equal to the diameter of the unrefracted image of the sun, s′, but its length is much greater.



Now if the solar beam consisted of rays possessing all the same degree of refrangibility, the refracted image should be circular, and of the same dimensions as the unrefracted image of the sun, s′, from which it would differ only in position. For the rays composing the beam, being parallel at their incidence on the prism, would, on this supposition, be equally refracted by it, and therefore continue parallel, and preserve the same mutual distances after refraction. This not being the case, then, we conclude that the rays composing the incident beam are of different degrees of refrangibility; the more refrangible rays going to form the upper part of the spectrum, v, and the less refrangible the lower, R.

In this account of the experiment we have assumed that the rays composing the solar beam are parallel. This, however, is not strictly the case; the rays proceeding from the upper and lower limbs of the sun being inclined to one another at an angle equal to the sun's apparent diameter, which is about half a degree. It will be easily seen, however, that this difference in the incidence of the rays which fall upon the prism is wholly insufficient to account for the phenomenon we have just described: for, if the prism be placed in its position of minimum deviation* with respect to the incident beam, a difference of incidence, such as that just mentioned, will produce no appreciable difference in the deviations of the emergent rays; consequently the rays will emerge from the prism inclined to one another at the same angle as at their incidence, if there be no difference in their degrees of refrangibility; and therefore the lengths of the refracted and unrefracted images will be the same, if received upon a screen at the same distance from the hole.

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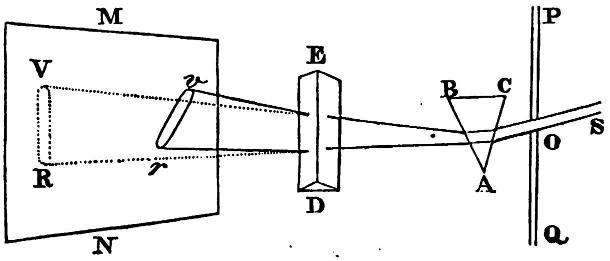
* This position is easily attained; for we have only to turn the prism slowly round its axis, and stop it when the spectrum, between its descent and subsequent ascent, appears stationary. This will be the position of minimum deviation (125.).

From the foregoing experiment it further appears that those rays, which differ in refrangibility, likewise differ in colour; the spectrum being red at its lowest or least refracted extremity R, violet at its most refracted extremity V, and yellow, green, and blue, in the intermediate spaces, these colours passing into one another by imperceptible gradations. Sir Isaac Newton, with the assistance of a person who had a more accurate eye than himself, distinguished the spectrum or coloured image of the sun into seven principal colours, and determined the spaces occupied by each. These colours, arranged in the order of their refrangibility, are *red, orange, yellow, green, blue, indigo, violet;* of which the yellow and orange are the most luminous, the red and green next in order, and the indigo and violet weakest.

Any one of these rays may be separated from the rest by transmitting it through a small aperture in a screen which intercepts the remainder of the light. The ray thus separated may be examined apart from the rest, and will be found to undergo no dilatation or change of colour by any subsequent refractions or reflexions. We are therefore warranted in concluding that the solar light is compound, and consists of an infinite number of simple rays, which are permanent in their own nature, but differ from one another both in their colour and refrangibility.

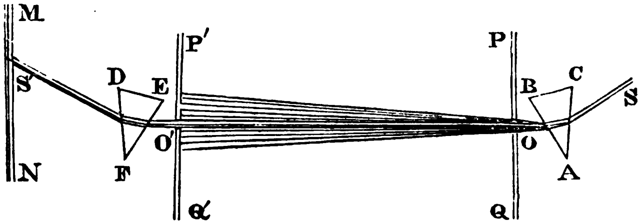
(232.) To the preceding conclusion, however, it may be objected that *each* ray of the incident beam is, by the action of the glass, dilated or split into many diverging rays without any difference in refrangibility; or, supposing a difference of refraction to exist, still that inequality may be *fortuitous*, and not the result of any real diversity in the rays themselves. The former of these hypotheses is that of Grimaldi, who was the first to observe the dilatation of the solar image by the prism.

To overthrow these objections the following experiment is adduced by Newton. The light emerging from the first prism, BAC, is received on a second, DE, whose axis is perpendicular to that of the first, and therefore the planes of their refractions perpendicular to one another. If, now, the length of the solar image arose from a dilatation of each ray, or from any casual inequality in their refractions, it is evident that the oblong image, RV, produced by the first refraction, should, by a second refraction made laterally, be increased in breadth, as it was before in length by the same causes, and should therefore appear square or rectangular. But the result is found to be otherwise: the image, *rv*, is not at all increased in breadth by the refraction of the second prism, but only becomes oblique to its former position, RV, the upper or violet extremity, *v*, being translated farther from its former position than the lower or red extremity, *r*. Accordingly, the light which is most refracted by the first prism is again most refracted by the second; and that which is least refracted by the first is, in like manner, least refracted by the second. And since the sides of the oblique image, *rv*, are found to be rectilinear, as well as those of the first, RV, it follows that every ray which is more or less refracted by the first prism, is, exactly in the same proportion, more or less refracted by the second.



Further, if the image produced by the second prism be again laterally refracted by a third, and so on to any number of refractions, it is always found that the rays which are more or less refracted by the first prism are in the same proportion more or less refracted by all the rest, and this without any dilatation of the image in breadth. These rays are therefore justly considered to possess each a peculiar degree of refrangibility.

(233.) The following experiment, however, may be considered as removing all doubt on this subject. Close behind the prism BAC is placed a board, PQ, perforated with a small aperture, O, through which the refracted light is permitted to pass: this light is then received on a second board, P′Q′, placed at a considerable distance from the first, and similarly perforated; so that a small portion of the light of the spectrum is suffered to pass through the aperture, O′, in the second board, the rest being intercepted. Close behind this aperture a second prism, DEF, is fixed, by which the transmitted light is a second time refracted. The first prism being then turned slowly round its axis, the light of the spectrum will move up and down on the board P′Q′, and the differently-coloured rays be successively transmitted through the aperture, O′, and fall upon the prism behind it. If then the places of the differently-coloured rays, S′, on the opposite wall, MN, be noted, the red will be found to be lowest, the violet highest, and the intermediate colours in order as they are in the spectrum. Here, on account of the unchanged position of the apertures in the boards, all the rays are necessarily incident upon the second prism, DEF, at the same angle; and yet, in that common incidence, some of them are more refracted and others less, and that in the same proportion as they are more or less refracted by the first prism. A ray of solar light consists, therefore, of an indefinite number of simple rays, each having its own degree of refrangibility.



Each of these simple rays, we have seen, has its peculiar colour, and these colours are permanent and unalterable in all the various modifications which the ray undergoes. For in the last-mentioned experiment it is found that, if the light be properly simplified, no perceptible change of colour is produced by the refraction of the second prism; and the same will be found true, if there be any number of successive refractions. Again, these colours undergo no change by reflexion; for it is found by experiment, that if any coloured body be placed in simplified homogeneous light, it will always appear of the colour of the light in which it is placed, whether that be the same with the colour of the body or not. Thus ultramarine and vermilion both appear red in a red light, blue in a blue light, &c. We shall return to this subject hereafter: at present it is noticed merely to show that simple homogeneous light suffers no change of colour by reflexion, whatever be the colour of the body by which it is reflected.

From the foregoing we conclude, then, that the peculiar colour and refrangibility belonging to each kind of homogeneous light are permanent and original affections not generated by the changes which that light undergoes in refractions or reflexions; and therefore that these properties are inherent in the rays previous to their separation by experiments.

(234.) In the experiments hitherto described, the analysis of solar light, or its resolution into its simple components, is far from being complete, inasmuch as there is a considerable mixture of the different species of simple light in the coloured image of the sun produced by the prism. This will be evident, if we consider that, as the several species of *homogeneous* light suffer no dilatation by the prism, each will depict on the screen a circular image of the sun equal in magnitude to the unrefracted image of the sun at the same distance. Hence the spectrum consists of innumerable circles of homogeneous light, whose centres are disposed along the same right line (the intersection, namely, of the plane of refraction with the screen), and whose common diameter is that of the unrefracted image. Wherefore the number of such circles mixed together in the spectrum is to the number mixed together in the unrefracted image of the sun, as the interval between the centres of two contingent circles, or the breadth of the spectrum, to the interval between the centres of the extreme circles, which is the length of the rectilinear sides. Now the mixtures being as the number of circles mixed together, we have

$$m' = m \cdot \frac{b}{l},$$

in which m' denotes the mixture of the different species of light in the spectrum, m, that in the unrefracted image, b the breadth of the spectrum, and l the length of its rectilinear side. The mixture, therefore, in the spectrum varies as $\frac{b}{l}$; if, therefore, b can be diminished, l remaining the same, the mixture will be diminished in proportion.

There are various ways of diminishing the breadth of the spectrum, or the diameter of the sun's unrefracted image, amongst which that of Newton seems as convenient in practice as can be required. In the beam of solar light, admitted into a darkened chamber through a small circular aperture, a lens of considerable focal length is placed at the distance of double its focal length from the aperture; at the same distance beyond the lens will be formed a distinct image of the hole, equal to it, which may be received upon a screen placed for that purpose. A prism being then placed immediately behind the lens, this image will be dilated in length, its breadth remaining unaltered; and thus a spectrum will be formed whose breadth is the diameter of the hole; whereas, without this contrivance, the breadth would be equal to that diameter together with a line which, at the distance of the screen from the hole, subtends an angle equal to the apparent diameter of the sun. Thus, by diminishing the diameter of the hole, the breadth of the spectrum, and therefore the mixture, may be reduced at pleasure.

If the diameter of the aperture be very small, the spectrum is reduced to a narrow line, which renders it unfit for examination. To remedy this inconvenience, Newton employed, instead of a circular aperture, a narrow rectangular one, whose length is parallel to the axis of the prism, and may be as large as we please, while its breadth is very small. In this manner we obtain a spectrum as broad as we wish, and whose light is, notwithstanding, as simple as before.

(235.) Having in this manner resolved the light of the sun into its simple and homogeneous components, the next step is to determine the degree of refrangibility belonging to each species.

It has been already mentioned (126.) that the index of refraction of any substance is given by the formula

$$\mu = \frac{\sin.\frac{1}{2}(\varepsilon + \theta)}{\sin.\frac{1}{2}\varepsilon};$$

in which ε denotes the refracting angle of the prism into which the substance is formed, and θ the minimum deviation of the ray, or the value which it has when the refractions are equal on both sides. This angle was obtained by Newton by measuring with a quadrant the altitude of the sun and the minimum inclination of the emergent ray to the horizon; and, if this could be done for each species of light of which the spectrum is composed, nothing more would be required. But this method, it is obvious, would not be susceptible of much accuracy in practice; and it will be found more convenient to determine the index of refraction of some determinate ray—the extreme red, for instance—by the preceding method, and, for the rest, to seek the relation between the corresponding small variations of μ and θ, the latter of which may be ascertained by observation.

Accordingly, if μ and θ denote the index of refraction and minimum deviation of the extreme red ray, $\mu + \delta\mu$ and $\theta + \delta\theta$ those of any other species, the relation between $\delta\mu$ and $\delta\theta$ will be obtained by differentiating the preceding equation, since the small corresponding variations of the quantities μ and θ may be regarded as proportional to the differentials of those quantities. Differentiating, therefore, with respect to the character δ, and dividing the result by the equation itself, we find

$$\delta\mu = \tfrac{1}{2}\mu \text{cotan.}\tfrac{1}{2}(\varepsilon + \theta)\delta\theta;$$

in which the quantity $\delta\theta$ is to be obtained by measuring the distance between the centre of the lower semicircular extremity of the spectrum and the ray whose index of refraction is sought: that interval divided by the distance from the point of emergence is the angle $\delta\theta$, which being substituted in the preceding equation, $\delta\mu$ is determined, and therefore $\mu + \delta\mu$ known.

(236.) The method employed by Newton consisted in determining the index of refraction of the extreme red and violet rays directly by means of the formula of the preceding article; and then, since $\delta\mu$ varies as $\delta\theta$, nearly, as appears from the formula just found, the index of refraction of the other rays may be obtained by a simple proportion.

Thus, if θ' and θ'' denote the minimum deviations of the extreme red and violet rays, and μ' and μ'' their refractive indices, we have

$$\mu' = \frac{\sin.\frac{1}{2}(\varepsilon + \theta')}{\sin.\frac{1}{2}\varepsilon}, \quad \mu'' = \frac{\sin.\frac{1}{2}(\varepsilon + \theta'')}{\sin.\frac{1}{2}\varepsilon}.$$

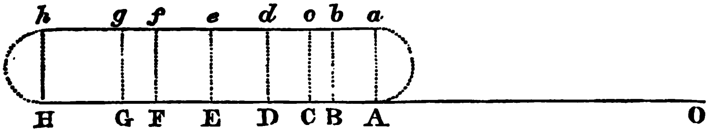
Now the difference of the angles θ' and θ'', which is the total dispersion, is obtained by dividing the interval between the centres of the extreme circles, or the difference between the length and breadth of the spectrum, by its distance from the prism. And the sum of these angles, which is double the deviation of the mean ray, is obtained by measuring the altitude of the sun and the inclination of the mean ray to the horizon at emergence. The sum and difference of the angles θ' and θ'' being thus obtained, the angles themselves are determined; and, their values being substituted in the preceding formulæ, the refractive indices of the extreme rays are known.

In this manner, when the refracting prism was of crown-glass, Newton found for the indices of refraction of the extreme red and violet rays

$$\mu' = \tfrac{77}{50}, \quad \mu'' = \tfrac{78}{50};$$

from which it follows that, for this medium, $\mu'' - \mu'$, or the whole variation of μ, is equal to $\frac{1}{50}$, or to ,02.

To determine the refractive indices of the several intermediate rays, it became necessary to examine the angles through which they were dispersed, and therefore the spaces which they occupied respectively in the coloured image of the sun. For this purpose Newton delineated on paper the spectrum A H *h a*, and distinguished it by the cross lines A *a*, B *b*, C *c*, &c. drawn



at the confines of the several colours; so that the space AB*ba* is that occupied by the red light, BC*cb* that by the orange, CD*dc* the yellow, DE*ed* the green, EF*fe* the blue, FG*gf* the indigo, and GH*hg* the violet. He then found that, if the whole length of the rectilinear side AH be taken as unit, the distances to the confines of the several colours AB, AC, AD, &c. will be denoted by the numbers $\frac{1}{8}$, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{9}$, 1; the refracting prism being of crown-glass, as before. Now the intervals AB, BC, CD, &c. occupied by the several colours in the spectrum, or the differences of the deviations which they subtend, are to one another as the corresponding variations of μ, the index of refraction. If, therefore, the whole variation of μ, or $\frac{1}{50}$, be divided as the line AH is in the points B, C, D, &c. the refractive indices of the rays at the confines of the several colours will be determined as follows:

$$\frac{77}{50}, \quad \frac{77\frac{1}{8}}{50}, \quad \frac{77\frac{1}{5}}{50}, \quad \frac{77\frac{1}{3}}{50}, \quad \frac{77\frac{1}{2}}{50}, \quad \frac{77\frac{2}{3}}{50}, \quad \frac{77\frac{7}{9}}{50}, \quad \frac{78}{50}.$$

The *mean* refractive index is $\frac{77\frac{1}{2}}{50}$, or 1,55; and it appears from the preceding that this belongs to the rays at the confines of the green and blue.

Newton observed that the rectilinear sides of the spectrum, AH and *ah*, were divided by the lines drawn at the confines of the several colours, after the manner of a musical chord. For, if HA be produced equally to O, the distances HO, GO, FO, EO, DO, CO, BO, AO, will be to one another as the numbers 1, $\frac{8}{9}$, $\frac{5}{6}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{9}{16}$, $\frac{1}{2}$, and so represent the length of the chords which sound the notes of the *octave*. This curious agreement, however, which Newton seems to have looked upon as the result of some physical relation, cannot be regarded in any other light than as an accidental and near coincidence.

(237.) On a minute examination of the solar spectrum, when every care has been taken in making the experiment, it is found that it is not, as Newton supposed, a *continuous* line of light, whose intensity is greatest about the confines of the yellow and orange, and diminishes regularly from thence to the two extremities; but that on the contrary there are, at certain points, abrupt deficiencies of light, total or partial, indicated by the existence of numerous dark streaks or *bands* crossing the spectrum in the direction of its breadth; and that, in the intermediate spaces, the intensity of the light does not increase or decrease continually, but varies irregularly, or according to some very complex law.

Some of these bands are wholly black, others dark, of various degrees of illumination: they differ also from one another in breadth, and are irregularly disposed throughout the length of the spectrum. They are not, however, the result of any accidental cause; for, when solar light is used, and the refracting substance of the same material, it is found that they preserve always the same relative breadth and intensity, the same relative position both with respect to one another and to the colours of the spectrum. When the refracting substance is varied, indeed, their relative positions *with respect to one another* are altered, but their positions as referred to the colours of the spectrum, as also their relative breadth and intensity, remain unchanged. These singular phenomena indicate that, in solar light, numerous rays of certain degrees of refrangibility are wanting, or at least that, if they do exist, they are not of a nature calculated to excite the sensation of vision.

If other kinds of light, as that of the fixed stars, flames, &c. are examined in the same way, similar bands are discovered, but differing in each species of light in their position, &c.; so that each species of flame and the light of each fixed star has its own system of bands, which remains unalterable under all circumstances, and which therefore is a distinct physical characteristic of the species of light to which it belongs.

These singular phenomena were first noticed by Wollaston, who published an account of his observations in the Philosophical Transactions of the year 1802. They have since been much more fully examined by the celebrated Fraunhofer, to whom the practical branches of optics and astronomy are so much indebted. Their exhibition requires the utmost nicety of observation, and a degree of purity in the refracting material which renders our ordinary glass prisms wholly inadequate to the purpose.

These bands, or *fixed lines*, are of the utmost importance in optical investigations, since, as they preserve an invariable position with respect to the colours of the spectrum, they furnish us as it were with landmarks by which the coloured image may be charted out and its proportions ascertained; and the accuracy of their delineation renders their observation susceptible of the utmost nicety.

In Newton's method of determining the refractive index of the different species of homogeneous light, already described, the regular gradation of tints in the coloured spaces of the spectrum makes it altogether impossible to fix with precision the position of any particular ray in the coloured image, and therefore to determine, with any accuracy, its index of refraction. The position of the fixed lines, however, may be observed with an accuracy equal to that of astronomical measurements, and thus the index of refraction of the rays, to which they correspond, determined with the utmost exactness. Fraunhofer fixed upon seven of the most remarkable of these lines, and determined the refractive indices of the rays corresponding to each for various media; by which the refractive powers of these media for these species of homogeneous light, as also their dispensive powers, are, with the utmost accuracy, determined.

(238.) As the white light of the sun is, by the foregoing experiments, resolved into its simple homogeneous elements, so these latter may be again compounded, and thus reproduce a white light agreeing in all its properties with the light directly received from the sun.

To effect this, Newton received the spectrum or coloured image of the sun upon a broad lens placed at the distance of double its focal length from the prism: by this means the coloured rays which diverge from the prism are by the lens made to converge to a focus, at the same distance behind the prism, and there by their mixture produce a perfect whiteness, as will be evident by receiving them on a white paper at that distance. If this paper or screen, upon which the light is received, be removed to a greater distance from the lens, the colours reappear in the contrary order, showing plainly that the whiteness at the focus is the result merely of the *mixture* of the variously-coloured rays, which converge to that point, and crossing there again diverge. The divergence of this beam may be corrected by means of another prism, whose refracting angle is equal to that of the former, placed at the focus where the rays are mixed. By this means the coloured rays, which after crossing at the focus would then diverge, are reduced to parallelism, and the emergent beam compounded of them is perfectly white. This compounded beam may be subjected to experiment, just as the direct light of the sun, and is found to be possessed of all the same properties; and if by a new refraction it be again resolved, and any of its component elements then stopped at the lens, the same colour will disappear from the refracted image; from which it is abundantly evident, that these several refractions produce no other effect than that of mixing or separating the rays of simple homogeneous light, without impressing any new modification on any, or changing in any respect their colorific qualities.

The coloured lights reflected from natural bodies may be in like manner compounded, and, if mixed together in the same proportion in which they enter solar light, will compose a whiteness which will be more or less perfect in proportion to the vividness of the colours employed.

Thus, if, by lines drawn from the centre, the area of a circle be divided in the proportion of the coloured spaces of the spectrum, and these sectors be painted with the prismatic colours, it is found that by a rapid rotation of this circle round an axis passing through the centre, the eye will be affected with the sensation of whiteness. Now, as the effect of any impression on the retina continues for some short time after the impression has been made, a rapid rotation will produce the same effect, as to vision, as if the several colours had been mixed together in the same object, and thus impressed simultaneously; and this effect, it is seen, is the sensation of whiteness.

It is to be observed, however, that all coloured bodies reflect the light of their own colours less copiously than white bodies do, as will be shown hereafter; and consequently, the combination of such reflected lights being less intense than that produced by the combination of the prismatic colours reflected by a white paper, the whiteness produced will be of a grayish hue, differing, not in quality, but degree, from the most intense whiteness.

CHAPTER II.

OF LIGHT DISPERSED BY REFRACTION AT PLANE SURFACES.

I.

Dispersion by a single Plane Surface, or by Prisms.

(239.) We have now seen that, though in the same refracting substance the index of refraction is invariable for any one species of homogeneous light, yet that, for each of the different species of such light of which the solar light is composed, it is different, and varies within certain limits dependent on the nature of the refracting substance; the less of these limiting values belonging to the extreme red ray of the spectrum, and the greater to the extreme violet. Accordingly, the fundamental equation of homogeneous refracted light, namely,

$$\sin.\theta = \mu . \sin.\theta',$$

may be extended to heterogeneous light by considering the index μ itself as a variable quantity, whose different values belong to the different species of simple light of which the compound ray is composed *.

(240.) In order to see, then, the variation in the position of the refracted ray arising from the diversity of the light of which the incident ray is compounded, we have only to consider the quantities μ and θ', in the preceding equation, as receiving any corresponding variations $\delta\mu$ and $\delta\theta'$; the resulting value of $\delta\theta'$ will be the quantity sought. Now these corresponding variations, being small, may be regarded without sensible error as proportional to the differentials of the quantities μ and θ'. Differentiating that equation, therefore, relatively to these quantities, and substituting $\delta\mu$ and $\delta\theta'$ for $d\mu$ and $d\theta'$, we find

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* In reflexion, considered as a case of refraction, $\mu = -1$, and is therefore the same for all kinds of light. There is, accordingly, no decomposition of solar light produced by reflexion, and therefore no separation into colours.

$$\delta\mu \,.\, \sin.\theta' + \mu\cos.\theta' \,.\, \delta\theta' = 0,$$

$$\therefore \delta\theta' = -\frac{\delta\mu}{\mu}\tan.\theta'.$$

When the course of the ray is from the denser into the rarer medium, θ is variable and θ' constant, and we find in like manner

$$\delta\theta = \frac{\delta\mu}{\mu}\tan.\theta.$$

Hence, if $\delta\mu$ be known for each species of ray, $\delta\theta$ or $\delta\theta'$ is found; and if its value be added to the angle of refraction of some one ray—as, for example, that of mean refrangibility—the angle of refraction, and therefore the direction of every other ray, is obtained.

When $\delta\mu$ is the whole variation of μ between its extreme limits, the corresponding value of $\delta\theta'$ is the measure of the *dispersion* which a ray of solar light undergoes in consequence of its separation by refraction at a plane surface. The quantities μ and $\delta\mu$, which enter its value, depend on the nature of the refracting medium, and are therefore constant for a given substance; the dispersion produced by refraction at the surface of a given medium, therefore, varies as the tangent of the angle of refraction.

(241.) In all known media, it has been observed, the absolute index of refraction is different for each of the different species of light of which solar light is composed, and therefore the quantity $\delta\mu$ is always of some finite magnitude. Hence, when the refraction takes place at the common surface of any medium and a *vacuum*, the dispersion $\delta\theta'$ can never vanish, unless when $\theta = 0$, or the ray incident perpendicularly on the surface; and accordingly the ray, when it suffers refraction at all, must also undergo a separation into its homogeneous elements.

When, however, the light is incident from one refracting medium into another of different density, these media may be so constituted that a ray of solar light may be refracted at their common surface without dispersion. For, if μ and μ' denote the absolute indices of refraction of the two media, the *relative index* of refraction at their common surface will be $\frac{\mu'}{\mu}$ (118.); and, that this should be constant for each of the different species of light of which the compound ray consists, we must have

$$\delta\left(\frac{\mu'}{\mu}\right) = 0, \quad \text{or, } \mu\delta\mu' - \mu'\delta\mu = 0;$$

$$i.\ e.\ \frac{\delta\mu'}{\delta\mu} = \frac{\mu'}{\mu}.$$

Wherefore, if the constitution of the two media be such that the increments of the refractive indices, for each of the different species of homogeneous light, are proportional to those indices themselves, the relative refractive index will be invariable, and the ray therefore will undergo refraction at the common surface of the two media without any separation into colours.

(242.) We now proceed to consider the dispersion of a ray of solar light, produced by refraction through a prism.

To solve the problem in all its generality, we shall suppose the ray dispersed before its incidence upon the prism, and seek the relation between the dispersions of the incident and emergent rays. Let ϕ and ψ, then, denote the angles which the portions of the ray in the rarer and denser medium respectively make with the perpendicular to the first surface at the point of incidence, ϕ' and ψ' the corresponding angles for the second surface, and ϵ the angle of the prism: the relations among these quantities are determined by the equations

$$\sin.\phi = \mu . \sin.\psi, \quad \sin.\phi' = \mu . \sin.\psi', \quad \psi + \psi' = \epsilon.$$

Wherefore, differentiating these equations regarding all the quantities, ε excepted, as variable, and substituting for the differentials of μ, ϕ, ψ, &c. the *whole* variations of these quantities between their extreme limits, which we shall denote by $\Delta\mu$, $\Delta\phi$, $\Delta\psi$, &c. we have

$$\cos.\phi\,.\,\Delta\phi = \Delta\mu\,.\,\sin.\psi + \mu\,.\,\cos.\psi\,.\,\Delta\psi$$

$$\cos.\phi'.\Delta\phi' = \Delta\mu\,.\,\sin.\psi' + \mu\,.\,\cos.\psi'.\,\Delta\psi'$$

$$\Delta\psi + \Delta\psi' = 0.$$

To eliminate $\Delta\psi$ and $\Delta\psi'$, we have only to multiply the first of these equations by $\cos.\psi'$, and the second by $\cos.\psi$, and add them together, and in virtue of the third we have

$$\cos.\psi'.\cos.\phi.\Delta\phi + \cos.\psi.\cos.\phi'.\Delta\phi' = \Delta\mu(\sin.\psi.\cos.\psi' + \cos.\psi.\sin.\psi');$$

or, since the coefficient of $\Delta\mu$ in the second member of this equation $= \sin.(\psi + \psi') = \sin.\,\varepsilon$,

$$\cos.\psi'\,.\,\cos.\phi\,.\,\Delta\phi + \cos.\psi\,.\,\cos.\phi'\,.\,\Delta\phi' = \Delta\mu\,.\,\sin.\varepsilon;$$

an equation which furnishes the relation between $\Delta\phi$ and $\Delta\phi'$, the dispersions of the incident and emergent rays.

(243.) When the ray is undispersed in its incidence on the prism, $\Delta\phi = 0$; and the dispersion of the emergent ray is

$$\Delta\phi' = \frac{\Delta\mu\,.\,\sin.\varepsilon}{\cos.\psi\,.\,\cos.\phi'}.$$

From which it appears that the dispersion varies inversely as the product of the cosines of the angles of refraction at the two surfaces of the prism.

When the ray is incident perpendicularly on the first surface of the prism, $\psi = 0$, and $\Delta\phi' = \frac{\Delta\mu.\sin.\varepsilon}{\cos.\,\phi'}$. But since $\psi = 0$, $\psi' = \varepsilon$, and therefore $\sin.\,\phi' = \mu\sin.\varepsilon$; and substituting in the preceding expression the value of $\sin.\varepsilon$, here obtained, there is

$$\Delta\phi' = \frac{\Delta\mu}{\mu}\tan.\phi',$$

as is otherwise evident from (240.), inasmuch as the ray suffers dispersion at the second surface only.

When the refractions are equal at both sides of the prism, $\psi = \psi' = \frac{\varepsilon}{2}$; therefore, substituting $\cos.\frac{\varepsilon}{2}$ for $\cos.\psi$ in the value of $\Delta\phi'$, $\Delta\phi' = \Delta\mu\,\frac{2\sin.\frac{1}{2}\varepsilon}{\cos.\phi'}$. But in this case there is $\sin.\phi' = \mu\,.\,\sin.\frac{1}{2}\varepsilon$; wherefore, dividing,

$$\Delta\phi' = \frac{2\Delta\mu}{\mu}\tan.\phi'.$$

(244.) From the general expression of $\Delta\phi'$ it is obvious that the dispersion, which a ray of solar light undergoes in passing through a prism, can never vanish, inasmuch as $\Delta\mu$ is always of finite magnitude, and the factors of the denominator are not capable of indefinite increase. It admits, however, of a minimum value, to which it attains when the denominator $\cos.\psi\,.\,\cos.\phi'$ is a maximum. Accordingly the condition of a minimum dispersion is obtained by differentiating the latter quantity, and making the result equal to nothing, and we find

$$\tan.\psi\,.\,d\psi + \tan.\phi'\,.\,d\phi' = 0.$$

If in this equation we substitute for $d\psi$ its value $-\,d\psi'$, it becomes

$$\tan.\phi'\,.\,d\phi' = \tan.\psi\,.\,d\psi';$$

and it only remains to eliminate $d\phi'$ and $d\psi'$ by means of the equation $\sin.\phi' = \mu\,.\,\sin.\psi'$: accordingly, if this equation be differentiated, and the result divided by the equation itself, there is

$$\text{cotan}.\phi'\,.\,d\phi' = \text{cotan}.\psi'\,.\,d\psi';$$

and, dividing the equation last found by this,

$$\tan.^2\phi' = \tan.\psi\,.\,\tan.\psi'.$$

The dispersion, therefore, is a minimum, when the tangent of the angle of emergence is a mean proportional between the tangents of the angles of refraction within the prism.

By this equation combined with the two following,

$$\sin.\phi' = \mu\,.\,\sin.\psi', \qquad \psi + \psi' = \varepsilon,$$

the position of minimum dispersion is completely determined.

To eliminate among these equations, let us square the equation $\sin.\phi' = \mu . \sin.\psi'$, and substitute in the result the value of the sine expressed in terms of the tangent, and it is

$$\frac{\tan.^2\phi'}{1+\tan.^2\phi'} = \frac{\mu^2 . \tan.^2\psi'}{1+\tan.^2\psi'};$$

and substituting in this for $\tan.^2\phi'$ its value $\tan.\psi . \tan.\psi'$, given by the equation of condition, we find

$$\mu^2 . \tan.\psi' = \frac{\tan.\psi(1+\tan.^2\psi')}{1+\tan.\psi . \tan.\psi'},$$

$$\text{or } (\mu^2 - 1)\tan.\psi' = \frac{\tan.\psi - \tan.\psi'}{1+\tan.\psi . \tan.\psi'},$$

subtracting $\tan.\psi'$ from both sides. We have, therefore, finally,

$$(\mu^2 - 1)\tan.\psi' = \tan.(\psi - \psi').$$

Since $\psi + \psi' = \varepsilon$, the second member of this equation is equivalent to $\tan.(\varepsilon - 2\psi')$; and, if this be developed in terms of $\tan.\psi'$, we shall obtain finally a cubic equation for the determination of the latter quantity *.

(245.) Returning to the general expression of the dispersion (243.), it will be easy to see in what manner it varies with the incidence.

In the first place, when the incident ray just grazes the surface, proceeding from the back towards the edge, $\phi = 90°$, and therefore ψ is greatest. Also $\psi' = \psi - \varepsilon$ is greatest, and

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* According to Newton, the position of minimum dispersion is the same as that of minimum deviation—that, namely, in which the refractions are equal at incidence and emergence. From the preceding determination, however, it is evident that this cannot be the case; since, if $\psi = \psi'$, we must have $\psi' = 0$, results which are inconsistent if the angle of the prism be of finite magnitude. Newton's error, if such it may be called (for the result is in strict conformity with his assumed data), arose from considering the ray to proceed out of the prism in both directions, and therefore the homogeneous rays to be parallel, not at their incidence on the prism externally, but within the prism itself. This, however, must be regarded only as a simplification of the real problem.

therefore likewise ϕ'. In this position, accordingly, $\cos.\psi.\cos.\phi'$ is a minimum, but finite; and therefore the dispersion is finite and a maximum.

As the angle which the incident ray makes with the side of the prism, towards the base, increases from nothing, ϕ and ψ are diminished, and consequently also ψ' and ϕ': accordingly, the denominator of the value of $\Delta\phi'$ increases, and the dispersion diminishes, until the denominator attains its maximum value, in which case the dispersion is the least possible. This position has been already discussed in the preceding article.

When the angle which the incident ray makes with the side of the prism towards the base still further increases, the dispersion increases indefinitely, since ϕ' increases, and therefore $\cos.\phi'$ decreases, without limit. Finally, when the emergent ray just grazes the surface, which is the limiting position at which a ray can be transmitted, $\cos.\phi' = 0$, and the dispersion becomes infinite. Since, therefore, the dispersion may be indefinitely increased by adjusting the position of the prism with respect to the incident ray, it follows that the dispersion produced by any prism, whose refracting angle is ever so great, may be counteracted by the dispersion of another prism, even of the same material, whose refracting angle is ever so small.

(246.) Let it be required to determine the dispersion of a ray produced by transmission through two prisms placed in any manner with respect to each other.

Let ε' be the refracting angle of the second prism, and μ' the index of refraction of the substance of which it is composed; also, let ϕ'' and ψ'', ϕ''' and ψ''' be the angles corresponding to ϕ and ψ, ϕ' and ψ' in the first prism: these angles are connected by the equations

$$\sin.\phi = \mu.\sin.\psi, \quad \sin.\phi' = \mu.\sin.\psi',$$

$$\sin.\phi'' = \mu'.\sin.\psi'', \quad \sin.\phi''' = \mu'.\sin.\psi''',$$

$$\psi + \psi' = \varepsilon, \quad \psi'' + \psi''' = \varepsilon'.$$

Also, if the inclination of the adjacent sides of the two prisms be denoted by ι, the angles ϕ' and ϕ'' are connected by the equation

$$\phi' + \phi'' = \iota.$$

Again, it appears from (242.) that the dispersions of the ray at its ingress into and egress from each prism are connected by the equations

$$\cos.\psi' . \cos.\phi . \Delta\phi + \cos.\psi . \cos.\phi' . \Delta\phi' = \Delta\mu . \sin.\varepsilon,$$

$$\cos.\psi''' . \cos.\phi'' . \Delta\phi'' + \cos.\psi'' . \cos.\phi''' . \Delta\phi''' = \Delta\mu' . \sin.\varepsilon' ;$$

and, since $\varphi' + \varphi'' = \iota$, we have also

$$\Delta\varphi' + \Delta\varphi'' = 0.$$

To eliminate among these equations, we have only to obtain the values of $\Delta\varphi'$ and $\Delta\varphi''$ from the two former, and substitute them in the latter, and the equation thence resulting will contain only $\Delta\varphi$ and $\Delta\varphi'''$, and therefore determine the relation between the dispersions of the incident and emergent rays.

When the ray is undispersed at its first incidence, $\Delta\varphi = 0$, and the resulting value of $\Delta\varphi'''$ is the dispersion produced by transmission through the two prisms.

(247.) From what has been said it will be readily seen in what manner we are to proceed in order to obtain the equations which determine the dispersion produced by any combination of prisms.

When the ray passes nearly perpendicularly through any number of prisms whose refracting angles are very small, the expression of the dispersion becomes very simple. For, in this case, the partial deviation produced by each prism is equal to the refracting angle multiplied by the index of refraction diminished by unity. Wherefore, if ε, ε', ε'', &c. denote the refracting angles of the prisms, μ, μ', μ'', &c. the refractive indices of the substances of which they are composed, and δ the total deviation of the ray, there is

$$\delta = (\mu - 1)\varepsilon + (\mu' - 1)\varepsilon' + (\mu'' - 1)\varepsilon'' + \&c.$$

and, differentiating with respect to δ, μ, μ', &c.

$$\Delta\delta = \Delta\mu . \varepsilon + \Delta\mu' . \varepsilon' + \Delta\mu'' . \varepsilon'' + \&c.$$

II.

Achromatic Combinations of Prisms.—Measurement of dispersive Powers.

(248.) It has been already shown (244.) that a ray of solar light cannot pass through a single prism without dispersion. If, however, a second prism be combined with the first, it is always possible to adjust their refracting angles in such a manner that the dispersion produced by the first shall be counteracted by the second, and consequently that the ray shall emerge colourless.

This Newton conceived to be impracticable, unless when the ray emerged in a direction parallel to that which it had at incidence, or the total deviation was nothing. He was led to this conclusion by observing that when a glass prism was enclosed in a prism of water with a variable refracting angle, their refracting angles being turned in opposite directions, the emergent ray was always coloured when it emerged inclined to its original direction, while, on the contrary, it was colourless whenever, by a proper adjustment of the angle of the water prism, it emerged parallel. From this he concluded that the dispersion of all substances was proportional to the refraction or deviation of the mean ray; and, therefore, that the dispersion could never be destroyed as long as any refraction took place. This made him despair of the improvement of refracting telescopes, and led him to turn his attention to the application of concave mirrors, as a substitute for the convex object-glasses of such instruments. Thus to his error on this subject we owe the invention of the telescope which goes by his name.

Dollond, an eminent London optician, repeated the experiments of Newton with the two prisms, and to his great surprise found the results exactly the opposite to those stated by Newton; the ray being coloured when it emerged parallel to its original direction, while on the other hand it was found inclined to its original direction when, by a proper adjustment of the angle of the water prism, it was made to emerge free from colour*. Thus it appeared that the dispersions produced by two prisms were equal while their refractions were unequal; and therefore that Newton's conclusion, that the dispersions were in all cases proportional to the mean refractions, was not consonant to experience.

(249.) But, further, it appears from an examination of the values of the index of refraction of different media for the different species of simple light, that not only are the whole spaces, over which any ray of solar light is dispersed, not proportional to the refraction of the mean ray, but also that the proportion of the spaces occupied by the several colours is different for different media. From this, which is termed the *irrationality* of the coloured spaces of the spectrum, it follows that, though the total dispersions produced by two prisms of different materials may be equal and opposite, and thus the extreme red and violet rays united in the emergent beam, yet there will be still a dispersion of the intermediate rays, the middle or green rays being more refracted, in proportion to the extremes, by one prism than by the other. Thus the ray, instead of emerging colourless from the two prisms, will form a second but smaller spectrum, one extremity of which is of a greenish tint, and the other of a colour compounded of the extreme red and violet rays, that is, of a purple hue. This is called the *secondary spectrum.*

Again, as secondary spectra arise when the extreme red and violet rays are united in the emergent beam by means of two prisms of different materials, so if three media be employed for the purpose of uniting three rays—as for instance the red, green, and violet—there will in like manner arise a *tertiary spectrum*, from the want of union of the other rays in the emergent beam; and so on indefinitely. But though in theory it is perhaps impossible to attain perfect achromatism by any combination of media, however numerous, yet it is evident that the successive spectra which arise will be much smaller and fainter, each than the preceding, so as after a few combinations to become wholly insensible; so much so, that in practice it is seldom deemed necessary to combine more than the two extreme rays.

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* Newton's mistake was detected, and the difference of the dispersive powers of different media first observed by Mr. Hall, a gentleman of Worcestershire, who applied his discovery to the construction of an achromatic telescope. This important discovery, however, was afterwards suffered to fall into oblivion, until it was again discovered and applied by Dollond. It is supposed that Newton's mistake arose from the accidental circumstance that the dispersive power, or the ratio of the dispersion to the mean refraction, of the species of glass which he employed was nearly the same as that of water.

It should be observed, however, that when two media are employed for the purpose of correcting the primary dispersion, the extreme red and violet rays, being faint, are not those which should be selected for union in the emergent beam; but we should rather combine those rays whose brightness and difference of colour together is greatest. Thus the near union of the other rays is better consulted, or the secondary spectrum less, than if we united the extreme rays.

(250.) When a ray of solar light is incident upon a combination of two prisms, it is required to determine the condition which they must fulfil, in order that the extreme red and violet rays should be united in the emergent beam, or the dispersion of the first order destroyed.

The extreme rays being united both in the incident and emergent beam, $\Delta\varphi = 0$, $\Delta\varphi''' = 0$, in the equations of (246.): these equations, therefore, are reduced to

$$\cos.\psi.\cos.\varphi'.\Delta\varphi' = \Delta\mu.\sin.\varepsilon,$$

$$\cos.\psi'''.\cos.\varphi''.\Delta\varphi'' = \Delta\mu'.\sin.\varepsilon',$$

$$\Delta\varphi' + \Delta\varphi'' = 0;$$

and, eliminating $\Delta\varphi'$ and $\Delta\varphi''$ from the two former by means of the third, we find

$$\Delta\mu.\sin.\varepsilon.\cos.\varphi''.\cos.\psi''' + \Delta\mu'.\sin.\varepsilon'.\cos.\varphi'.\cos.\psi = 0.$$

It appears from (246.) that the angles φ' and ψ, φ'' and ψ''', are connected by the equations

$$\sin.\varphi' = \mu.\sin.(\varepsilon - \psi), \qquad \sin.\varphi'' = \mu'.\sin.(\varepsilon' - \psi''');$$

so that, μ and μ' being given, there are four quantities in the equation of condition really independent, namely, ε and ε', ϕ and ϕ''. Hence, if the angle of the first prism be given, as well as its position with respect to the incident ray, ε and ϕ' are given, and ε' and ϕ'' remain arbitrary. Accordingly, if one of these quantities be assumed as fixed, the equation of condition may be satisfied by the other; from which it follows that the combination may be rendered achromatic in two ways—either by varying the angle of the second prism, its position being given, or by varying its position when its angle is given.

These results are true, whatever be the values of μ and μ', $\Delta\mu$ and $\Delta\mu'$; it appears therefore that, even in the case in which the two prisms are of the same material, the emergent beam may be achromatized in either way.

If the prisms are both placed in the position of minimum deviation, the equation which furnishes the condition of achromatism assumes a very simple form. For, since in this case the refractions are equal at both sides, there is

$$\psi = \psi' = \frac{\varepsilon}{2}, \qquad \psi'' = \psi''' = \frac{\varepsilon'}{2};$$

and, these values being substituted in the equation obtained above, it becomes

$$\Delta\mu \,.\, \sin.\tfrac{1}{2}\varepsilon \,.\, \cos.\,\phi'' + \Delta\mu' \,.\, \sin.\tfrac{1}{2}\varepsilon' \,.\, \cos.\,\phi' = 0;$$

which, if we substitute for $\sin.\tfrac{1}{2}\varepsilon$, $\sin.\tfrac{1}{2}\varepsilon'$, their values derived from the equations

$$\sin.\,\phi' = \mu \,.\, \sin.\tfrac{1}{2}\varepsilon, \qquad \sin.\,\phi'' = \mu' .\, \sin.\tfrac{1}{2}\varepsilon',$$

is reduced to

$$\mu'\Delta\mu \,.\, \tan.\,\phi' + \mu\Delta\mu' \,.\, \tan.\,\phi'' = 0,$$

in which ϕ' and ϕ'' are determined by the two preceding equations.

(251.) There would be no difficulty in extending these investigations so as to determine the condition of achromatism of a ray of solar light which is transmitted through any combination of prisms. But the results are complicated and of little practical importance.

There is one case, however, in which the condition of achromatism assumes a very simple form: it is that in which a ray passes nearly perpendicularly through any combination of prisms whose refracting angles are small. For, equating to nothing the value of the dispersion in this case (247.), we have

$$\Delta\mu . \varepsilon + \Delta\mu' . \varepsilon' + \Delta\mu'' . \varepsilon'' + \&c. = 0.$$

(252.) The dispersive power of a prism is measured by the difference of the deviations of the extreme red and violet rays divided by the total deviation. Now, when the refracting angle of the prism is small, and the ray incident upon it nearly perpendicularly, this fraction is independent both of the angle of the prism and of the incidence of the ray upon it, and is a function of the index of refraction only. This function, therefore, depending solely upon the nature of the medium of which the prism is formed, becomes the natural measure of the dispersive power of that substance. To determine its value we have (124.)

$$\delta = (\mu - 1)\varepsilon, \quad \text{and } \Delta\delta = \Delta\mu . \varepsilon;$$

in which ε denotes the refracting angle of a thin prism, and δ the deviation of a ray which passes through it nearly perpendicularly. Dividing, then, the latter of these equations by the former, there is

$$\frac{\Delta\delta}{\delta} = \frac{\Delta\mu}{\mu - 1}.$$

The *dispersive power* of any substance, therefore, is measured by the quantity $\frac{\Delta\mu}{\mu - 1}$, in which $\Delta\mu$ denotes the whole variation of the index of refraction between its extreme limits.

Its magnitude may be ascertained by forming the substance, whose dispersive power is required, into a prism, and ascertaining by direct measurement the dispersion which a ray of solar light undergoes in passing through it. This is to be obtained by receiving the spectrum perpendicularly upon a screen at a sufficient distance, and dividing the length of its rectilinear sides, or the difference between the length and breadth of the spectrum, by its distance from the prism. The magnitude of the dispersion being thus found, and the relation between it and the increment of the refractive index known (243.), the latter is then determined, and therefore the dispersive power.

Thus, if the prism be placed in the position of minimum deviation, from (243.) we find

$$\frac{\Delta\mu}{\mu} = \tfrac{1}{2}\frac{\Delta\varphi'}{\tan.\varphi'}, \quad \therefore \frac{\Delta\mu}{\mu - 1} = \tfrac{1}{2}\frac{\mu}{\mu - 1}\cdot\frac{\Delta\varphi'}{\tan.\varphi'};$$

in which $\Delta\varphi'$ is found by the method just described, and $\tan.\varphi'$ determined by the equation

$$\sin.\varphi' = \mu.\sin.\tfrac{1}{2}\varepsilon,$$

ε being the angle of the prism. The dispersive power of any substance, however, is more conveniently ascertained in practice by comparing it with that of some other substance in which it is known.

(253.) In order to compare together the dispersive powers of two substances, they are to be formed into prisms, and the combination rendered *achromatic* either by varying the angle of the second prism, or by changing its position with respect to the first. This being done, the equation which expresses the condition of achromatism will give the ratio $\frac{\Delta\mu}{\Delta\mu'}$, and therefore, if the refractive indices of the two substances be already known, the ratio of their dispersive powers.

The method usually adopted in practice is to adjust the angle of the second prism. In this method, therefore, a prism of some known substance (such as glass or water) is required, whose refracting angle is capable of being varied at pleasure. The two prisms being then fixed in any position (that which gives most simplicity to the result is that in which the refractions are equal at both sides), the angle of the variable prism is increased or diminished until the ray emerges colourless. This is determined by looking through the prisms at any dark object upon a white ground (as for example one of the bars of a window-frame), and observing when the coloured fringes, by which such an object is usually surrounded, disappear. This being done, and the angle of the variable prism then measured, the ratio of the dispersive powers of the two prisms is determined by the equation of achromatism. Thus, when the two prisms are placed in the position of least deviation, that equation (250.) gives

$$\frac{\Delta\mu}{\Delta\mu'} = -\frac{\mu}{\mu'} \cdot \frac{\tan.\varphi''}{\tan.\varphi'};$$

and therefore the ratio of the dispersive powers is

$$\frac{\Delta\mu}{\Delta\mu'} \cdot \frac{\mu' - 1}{\mu - 1} = -\frac{\mu}{\mu'} \cdot \frac{\mu' - 1}{\mu - 1} \cdot \frac{\tan.\varphi''}{\tan.\varphi'},$$

in which the angles φ' and φ'' are determined by the equations

$$\sin.\varphi' = \mu . \sin.\frac{\varepsilon}{2}, \qquad \sin.\varphi'' = \mu' . \sin.\frac{\varepsilon'}{2}.$$

When the angles of the prisms are small, the ratio of the angles φ' and φ'', ε and ε', may be substituted for that of their sines or tangents; wherefore, from the preceding equations, there is

$$\frac{\varphi''}{\varphi'} = \frac{\mu' . \varepsilon'}{\mu . \varepsilon};$$

and, this value being substituted for $\frac{\tan.\varphi''}{\tan.\varphi'}$ in the above value of the ratio, it becomes

$$\frac{\Delta\mu}{\Delta\mu'} \cdot \frac{\mu' - 1}{\mu - 1} = -\frac{(\mu' - 1)\varepsilon'}{(\mu - 1)\varepsilon}.$$

That is, the dispersive powers of the two prisms must be inversely as the total deviations, as may readily be shown independently.

For the practice of this method, it is evident from what has been said, it becomes necessary that we should be provided with a prism of some known substance, whose refracting angle is capable of being varied at pleasure. The most obvious method of constructing such a prism is to enclose water or any other fluid in a vessel composed of two plates of glass with parallel surfaces united by a hinge round which they turn; the sides of the vessel being closed by metallic cheeks, which fit in such a manner as to prevent the escape of the fluid. Clairaut employed for this purpose a plano-cylindrical lens, the different parts of whose cylindrical surface contained of course every possible angle with the plane surface from 0 to 90°. Boscovich improved considerably upon this contrivance by joining together two such plano-cylindrical lenses, one of which was plano-convex and the other plano-concave, and of equal curvatures. The convex surface then fitting in the concave, and revolving round the common axis of the two cylinders, it is obvious that the plane surfaces will be inclined at every possible angle.

(254.) Each of these methods, however, of varying the angle of the standard prism is found liable to some important objection in practice; and to avoid such objections Dr. Brewster proposed a very ingenious contrivance, in which, by altering the *plane of refraction* of the standard prism, the same effect is produced as if its *angle* had been variable. The change in the plane of refraction by which the achromatic adjustment is effected then determines the ratio of the dispersive powers of the two prisms. In this manner Dr. Brewster has calculated the dispersive powers of a great number of substances*.

From what has been said above, however, it is plain that it is not necessary to resort to this expedient; since it has been shown that by simply changing the *inclination* of the second prism to the first, its plane of refraction as well as its angle remaining unaltered, it is possible to correct the dispersion produced by the first. This being done, the equation which expresses the condition of achromatism gives the ratio of the quantities $\Delta\mu$ and $\Delta\mu'$, and therefore the ratio of the dispersive powers.

In this equation, if one of the angles, as ψ, be known, the rest are determined by the equations (246.). The value of ψ is determined by the position of the first prism, which is perfectly arbitrary, and the position which seems to give the most simplicity to the result is that in which the first surface of this prism is perpendicular to the incident ray: in this case $\psi = 0$, and $\psi' = \varepsilon$, and the equation of achromatism (250.) gives

$$\frac{\Delta\mu}{\Delta\mu'} = -\frac{\sin.\varepsilon'}{\sin.\varepsilon} \cdot \frac{\cos.\varphi'}{\cos.\varphi''.\cos.\psi'''};$$

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* See Appendix.

in which the angles ϕ', ϕ'', and ψ''', are given by the equations

$$\sin.\phi' = \mu.\sin.\varepsilon, \qquad \sin.\phi'' = \mu'.\sin.\psi'',$$
$$\phi' + \phi'' = \iota, \qquad \psi'' + \psi''' = \varepsilon'.$$

Accordingly, the angles ε and ε' being given, the adjustment is to be performed by the variation of the angle ι, or the inclination of the adjacent surfaces of the two prisms; and the adjustment being performed, and the angle ι then ascertained by direct observation, the angles ϕ', ϕ'', and ψ''', are obtained by calculation from the equations just given.

It is obvious that the angle ι may be ascertained by the very process of making the achromatic adjustment. For, if the two prisms be attached to a frame, the first invariably, and the second by means of an axis round which it revolves, it is evident that an index, attached to the extremity of the axis of the revolving prism, will revolve through an angle equal to that which the side of this prism contains with its first position; and therefore will mark off on a graduated arch attached to the frame an angle equal to the angle formed by the adjacent surfaces of the two prisms, provided that it points to the *zero* of the scale when these surfaces are parallel.

CHAPTER III.

OF LIGHT DISPERSED BY REFRACTION AT SPHERICAL SURFACES.

I.

Dispersion by a Single Lens, or by any Combination of Lenses.

(255.) We have seen (156.) that the power of a lens is expressed by the formula

$$\varphi = (\mu - 1)(\varrho - \varrho'),$$

in which ϱ and ϱ' denote the curvatures of the two surfaces. Now it is evident that this quantity, being a function of μ, the index of refraction, will be different for each of the different species of homogeneous light; being greatest for the violet or most refrangible rays, least for the red or least refrangible, and of intermediate values for the rest, in the order of their refrangibility. Accordingly, when a pencil of parallel rays of solar light is incident upon a lens, the various simple rays which compose it will, after refraction, converge to or diverge from different foci; the focus of the violet rays being nearest to the lens, that of the red farthest from it, and those of the intermediate rays occupying intermediate positions.

This difference of the vergencies of the different species of simple light is easily exhibited. Let a sheet of white paper be placed so as to receive perpendicularly the cone of light converging to the focus of a convex lens. When the paper is brought nearer to the lens than the focus of the rays of mean refrangibility, the circle formed by the intersection of this cone with the paper will be fringed with *red*; and, if placed beyond that focus, with *blue*: for the red rays, having the least convergency, will be outermost within the focus, and innermost beyond it; while, on the contrary, the violet rays will be innermost within the focus, and outermost beyond it.

(256.) In order to see the law of this variation, let μ and φ denote the refractive index and power of the lens for the rays of least refrangibility, $\mu + \delta\mu$ and $\varphi + \delta\varphi$ those for any other species of rays; then, if the latter quantities be substituted in the preceding equation, and that equation subtracted from the result, we have

$$\delta\varphi = \delta\mu(\varrho - \varrho');$$

and, dividing this by the equation itself,

$$\delta\varphi = \frac{\delta\mu}{\mu - 1}\varphi.$$

When $\delta\mu$ is equal to $\Delta\mu$, the whole variation of the refractive index between its extreme limits, $\delta\phi$ becomes $\Delta\phi$, the corresponding variation of ϕ, and therefore

$$\Delta\phi = \frac{\Delta\mu}{\mu - 1}\phi;$$

that is, the whole variation of the power of a lens, arising from the heterogeneity of the light, is equal to that power itself multiplied by the dispersive power of the substance of which it is formed.

When the incident rays diverge from a point, the relation between the vergencies of the incident and refracted pencils is expressed by the equation

$$\alpha' - \alpha = \varphi.$$

And if, for any other rays of the incident pencil, α' and φ become $\alpha' + \delta\alpha'$ and $\varphi + \delta\varphi$ respectively, substituting and subtracting, we have

$$\delta\alpha' = \delta\varphi.$$

That is, the variation of the vergency of any refracted pencil is altogether independent of the vergency of the incident pencil, and equal to the variation of the power of the lens.

(257.) From what has been said above, it appears that the different species of simple light, which are united in the incident pencil, will, after refraction, diverge from or converge to different points in the axis of the lens. The space, over which these foci of the differently coloured rays are diffused, is called the *chromatic aberration.* After what has been said its magnitude is easily ascertained; for, if f denote the principal focal length of the lens,

$$\phi = \frac{1}{f}, \quad \text{and } \Delta\phi = \frac{-\Delta f}{f^2};$$

since the variations of ϕ and f, being small, may without sensible error be regarded as proportional to the differentials of these quantities. And substituting in the equation of the preceding article

$$\Delta f = -\frac{\Delta\mu}{\mu - 1} f;$$

that is, the chromatic aberration from the principal focus is equal to the focal length multiplied by the dispersive power.

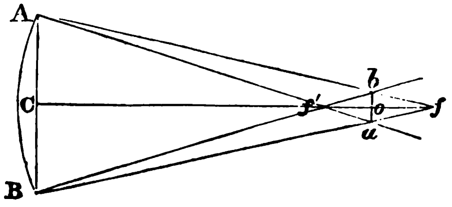
In like manner, if d be the distance of the focus of any refracted pencil from the lens, $\alpha' = \frac{1}{d}$, and $\Delta\alpha' = -\frac{\Delta d}{d^2}$; wherefore, since $\Delta\alpha' = \Delta\phi$, there is

$$\Delta d = \frac{d^2}{f^2}\Delta f.$$

That is, the aberration from the conjugate focus is to that from the principal focus in the duplicate ratio of the focal distances themselves. If in this result we put for Δf its value found above, we have

$$\Delta d = -\frac{\Delta\mu}{\mu - 1} \cdot \frac{d^2}{f}.$$

(258.) We have now seen that when a pencil of solar rays is refracted by a lens, the different simple lights of which it is composed will be separated by the refraction, and diverge from, or converge to, each a different focus. Hence, even supposing that no aberration is produced by the form of the refracting surfaces, the refracted light is nowhere brought to a point; and, accordingly, it is naturally suggested to us to inquire the magnitude and position of the least possible space into which the refracted rays are collected. This space is called the *least circle of chromatic aberration.*



Let Af, Bf, be the extreme red rays of a pencil refracted by the lens AB; Af', Bf', the extreme violet intersecting the former in a and b; then it is plain that the connecting line ab is the diameter of the least circle into which all the rays are collected, or the diameter of the least circle of chromatic aberration.

Now, in the similar triangles AfB and afb, Af'B and $af'b$, there is

$$of = cf.\frac{ab}{AB}, \qquad of' = cf'.\frac{ab}{AB};$$

$$\therefore\ of + of' = \frac{ab}{AB}(cf + cf').$$

Now $of + of' = ff' = cf - cf'$; therefore there is

$$ab = AB.\frac{cf - cf'}{cf + cf'}.$$

But $cf - cf'$, the difference of the focal lengths of the extreme rays, is the aberration Δd; and $cf + cf'$ is equal to double the focal length of the mean rays, or to $2d$. Wherefore, denoting the radius of the least circle of aberration by ϱ, and the semi-aperture by A,

$$2\varrho = A\frac{\Delta d}{d} = A\frac{\Delta\mu}{\mu - 1}.\frac{d}{f}.$$

When the incident rays are parallel, $d = f$; and therefore

$$2\varrho = A\frac{\Delta\mu}{\mu - 1}.$$

That is, the diameter of the least circle of chromatic aberration for parallel rays is equal to the semi-aperture of the lens multiplied by the dispersive power. For a given substance, therefore, it varies simply as the aperture.

In crown-glass $\frac{\Delta\mu}{\mu - 1} = ,033$; and therefore the diameter of this circle in the principal focus of a lens of this substance $= ,033 . \text{A}$, or about the $\frac{1}{30}$th part of the aperture.

To find the position of the centre of the least circle of chromatic aberration, let the equations found at the commencement of this article be divided, one by the other, and we have

$$\frac{cf}{cf'} = \frac{of}{of'} = \frac{cf - co}{co - cf'};$$

from which we obtain

$$co = \frac{2cf . cf'}{cf + cf'}.$$

That is, the distance of the centre of this circle from the lens is a harmonic mean between the focal distances of the extreme rays.

Now, if d be the focal distance of mean rays, $cf = d + \frac{1}{2}\Delta d$, $cf' = d - \frac{1}{2}\Delta d$; and substituting these values in the expression of co, we have

$$co = d - \frac{(\Delta d)^2}{4d}.$$

Hence the distance of the centre of the least circle of chromatic aberration from the focus of mean rays is equal to $\frac{(\Delta d)^2}{4d}$.

(259.) The angular dispersion is equal to the diameter of the circle of aberration divided by its distance from the lens, or the conjugate focal distance. It is therefore equal to

$$\frac{2\varrho}{d} = \frac{\Delta\mu}{\mu - 1} . \frac{\text{A}}{f},$$

and is therefore independent of the position of the radiant point.

From this it is easy to discover the ratio of the focal lengths of two lenses which will counteract each other's dispersions. For, that the dispersion of the compound lens should be destroyed, the angular dispersions of the component lenses must be equal and opposite. Now these dispersions are as the dispersive powers of the substances of which the lenses are composed divided by their focal lengths, the apertures of the lenses being the same. It follows, therefore, that the focal lengths of the two lenses must be to one another as their dispersive powers. We shall presently consider this subject in a more general point of view.

(260.) Something is usually said by optical writers on the variation of the density of the light in the circle of chromatic aberration. To understand this, we are to consider that the several pencils of homogeneous light which are incident upon the lens, being brought each to a different focus by refraction, may be considered as forming so many cones of rays having a common base and axis, namely, the surface and axis of the lens; and whose vertices are arranged along their common axis in the interval between the foci of the extreme rays. These cones of light, therefore, if received upon a screen at the place of the least circle of aberration, will depict there so many concentric circles, whose radii diminish from the radius of the circle of aberration to nothing, and then increase to their former limit; and therefore the circle of aberration may be conceived to be composed of an indefinite number of such circles successively superimposed.

Now, if x and $x + dx$ denote the radii of two such contiguous circles, the change of density in proceeding from one periphery to the next will be evidently the density in the last or superimposed circle. But, if the quantity of light in these several circles be supposed equal, the density in each circle will be inversely as its area, or the square of its radius. Wherefore, if D and $\text{D} + d\text{D}$ be the densities in the circles whose radii are x and $x + dx$, $d\text{D}$ varies inversely as x^2, or

$$d\text{D} = \frac{m dx}{x^2},$$

m denoting an unknown constant; and integrating this equation, $\text{D} = -\frac{m}{x} + \text{const.}$ Now the density is nothing at the periphery, or $\text{D} = 0$ when $x = r$; hence the constant is equal to $+\frac{m}{r}$, and the corrected integral is

$$\text{D} = m\left(\frac{1}{r} - \frac{1}{x}\right).$$

The density therefore varies as $\frac{r-x}{x}$, *i. e.* as the distance from the periphery divided by the distance from the centre. Hence the light in this circle diminishes rapidly from the centre to the circumference, where it becomes extremely faint and at last vanishes; and, for this reason, the confusion arising from chromatic aberration is not so great as would appear from the consideration of its magnitude.

It is necessary to observe, however, that the preceding conclusions are built upon the assumption that the simple rays of each different degree of refrangibility exist all in the same quantity in solar light. This assumption the observations of the spectrum show to be without foundation in fact; and therefore the results that have been obtained above are only to be regarded as loose approximations.

(261.) We now proceed to consider the dispersion produced by any combination of lenses.

To commence with the simplest case, that, namely, in which the lenses composing the system are in contact: let φ', φ'', φ''', &c. denote the powers of the component lenses, and φ that of the system, for the rays of least refrangibility; $\varphi' + \delta\varphi'$, $\varphi'' + \delta\varphi''$, $\varphi''' + \delta\varphi'''$, &c. and $\varphi + \delta\varphi$, the corresponding quantities for the rays of any other species. Then, from (167.) it appears that

$$\varphi = \varphi' + \varphi'' + \varphi''' + \&c.$$

$$\varphi + \delta\varphi = (\varphi' + \delta\varphi') + (\varphi'' + \delta\varphi'') + (\varphi''' + \delta\varphi''') + \&c.$$

and subtracting,

$$\delta\varphi = \delta\varphi' + \delta\varphi'' + \delta\varphi''' + \&c.$$

And, if we substitute for $\delta\phi'$, $\delta\phi''$, &c. their values as obtained (256.), the variation of the power of the system is

$$\delta\phi = \frac{\delta\mu'}{\mu'-1}\phi' + \frac{\delta\mu''}{\mu''-1}\phi'' + \frac{\delta\mu'''}{\mu'''-1}\phi''' + \&c.$$

Such is the variation of the power in proceeding from the lowest or least refrangible rays to those of any other species. To obtain the whole variation of this quantity between its extreme limits, we have only to substitute for $\delta\mu'$, $\delta\mu''$, &c. the whole variations $\Delta\mu'$, $\Delta\mu''$, &c. Thus the coefficients of ϕ', ϕ'', &c. in the preceding expression become the dispersive powers of the substances of which the several lenses are composed; and if these be denoted by π', π'', π''', &c. we have

$$\Delta\phi = \pi'\phi' + \pi''\phi'' + \pi'''\phi''' + \&c.$$

When the rays diverge from a point, we have (167.), as in the case of a single lens,

$$\alpha' - \alpha = \phi,$$

in which α and α' denote the vergencies of the incident and refracted pencils respectively. And from this we find, as in a single lens,

$$\delta\alpha' = \delta\phi, \text{ and } \Delta\alpha' = \Delta\phi.$$

(262.) We shall now investigate the inequality arising from the heterogeneity of light in refraction through any number of lenses, disposed in any manner along the same axis.

The relations between the vergencies of the incident and refracted pencils are given by the equations

$$\beta = \alpha + \varphi, \quad \beta' = \alpha' + \varphi', \quad \beta'' = \alpha'' + \varphi'', \ \&c.$$

$$\frac{1}{\alpha'} = \frac{1}{\beta} + \frac{1}{\theta}, \quad \frac{1}{\alpha''} = \frac{1}{\beta'} + \frac{1}{\theta'}, \ \&c.$$

in which φ, φ', φ'', &c. denote the powers of the several lenses; α, α', α'', &c. the vergencies of the pencils incident on each; β, β', β'', &c. those of the refracted pencils; and θ, θ', θ'', &c. the reciprocals of the intervals between the lenses (168.). Now, the corresponding variations of these quantities arising from the heterogeneity of light, being small, may be regarded as proportional to their differentials. Therefore, differentiating the preceding equations, and substituting these variations for the differentials, we have

$$\delta\beta = \delta\varphi, \quad \delta\beta' = \delta\alpha' + \delta\varphi', \quad \delta\beta'' = \delta\alpha'' + \delta\varphi'', \ \&c.$$

$$\frac{\beta^2}{\alpha'^2}\,\delta\alpha' = \delta\beta, \quad \frac{\beta'^2}{\alpha''^2}\,\delta\alpha'' = \delta\beta', \ \&c.$$

To eliminate among these equations, let the second equation of the first series be multiplied by $\frac{\beta^2}{\alpha'^2}$, the third by $\frac{\beta^2}{\alpha'^2} \cdot \frac{\beta'^2}{\alpha''^2}$, &c. and let them be added together; then, in virtue of the relations furnished by the second series of equations, the quantities $\delta\beta$, $\delta\alpha'$, $\delta\beta'$, $\delta\alpha''$, &c. disappear from the result, and we obtain

$$\delta\beta^{(n)}\left(\frac{\beta\beta'\beta''}{\alpha'\alpha''\alpha'''}, \&c.\right)^2 = \delta\varphi + \delta\varphi'\left(\frac{\beta}{\alpha'}\right)^2 + \delta\varphi''\left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 + \&c.$$

in which it only remains to substitute for $\delta\varphi$, $\delta\varphi'$, &c. their values (256.).

For the whole variation of the vergency of the refracted pencil, we have

$$\delta\varphi = \Delta\varphi = \pi\varphi, \qquad \delta\varphi' = \Delta\varphi' = \pi'\varphi', \ \&c.$$

and therefore,

$$\Delta\beta^{(n)}\left(\frac{\beta\beta'\beta''}{\alpha'\alpha''\alpha'''}, \&c.\right)^2 = \pi\phi + \pi'\phi'\left(\frac{\beta}{\alpha'}\right)^2 + \pi''\phi''\left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 + \&c.$$

When the lenses are in contact, the fractions $\frac{\beta}{\alpha'}$, $\frac{\beta'}{\alpha''}$, $\frac{\beta''}{\alpha'''}$, &c. become each equal to unit, and the equation is reduced to that of the preceding article.

II.

Achromatic Combinations of Lenses.

(263.) After what has been said in the concluding part of the preceding section, the reader will find no difficulty in determining the condition which must be fulfilled by any combination of lenses, in order that a pencil of rays passing through them may be undispersed at its emergence, or the combination achromatic.

Thus, when a pencil of rays is refracted by any combination of lenses in contact, in order that the extreme red and violet rays should be united in the emergent pencil, the whole variation of the vergency of the refracted pencil arising from the heterogeneity of light must be nothing; that is, we must have (261.)

$$\Delta\alpha' = \Delta\phi = 0.$$

Whence, substituting for $\Delta\phi$ its value obtained in the same article, the condition of achromatism is

$$\pi'\phi' + \pi''\phi'' + \pi'''\phi''' + \&c. = 0.$$

This result, being independent of the vergency of the incident pencil, shows that if a compound lens, consisting of any number of simple lenses in contact, be achromatic for any one distance of the radiant, it will be so for every distance.

If any of the lenses thus combined be of the same material, their dispersive powers are the same; and that part of the first member of the preceding equation which relates to them becomes $\pi\Sigma(\phi)$, $\Sigma(\phi)$ denoting the sum of their powers. Accordingly, their effect as to achromatism is the same as that of the single equivalent lens, whose power is $\Sigma(\phi)$.

(264.) To apply the preceding theory to the case of two lenses in contact, the equation of achromatism becomes

$$\pi'\phi' + \pi''\phi'' = 0.$$

Wherefore, since π' and π'' are essentially positive, it follows that ϕ' and ϕ'' must be of opposite signs, or that one of the lenses must be *convex* and the other *concave;* and that their powers of refraction must be inversely, or their focal lengths directly, as the dispersive powers of the substances of which they are composed.

The preceding equation, combined with that which expresses the relation between the power of the compound and those of the component lenses, will determine the latter when the former is given. These equations being independent of the absolute magnitude of π' or π'', and involving their ratio only, it will be found convenient to introduce that ratio into the preceding result. Wherefore, dividing by π'', and denoting the ratio $\frac{\pi'}{\pi''}$ by p, we have

$$\varpi'\varphi' + \varpi''\varphi'' = 0, \quad \varphi' + \varphi'' = \Phi;$$

whence [illegible]

$$\varphi' = \frac{\Phi}{1-p}, \quad \varphi'' = \frac{p.\Phi}{p-1};$$

by which the powers of the component lenses are determined when the ratio of their dispersive powers and the power of the compound are known.

Thus, in the common double achromatic lens, in which the component lenses are of crown-glass and flint-glass, $\varpi' = ,033$ and $\varpi'' = ,05$, nearly; therefore $p = \frac{33}{50}$, and, consequently,

$$\varphi' = \frac{50}{17}\Phi, \quad \varphi'' = -\frac{33}{17}\Phi;$$

which are nearly 3Φ and -2Φ, respectively. From the signs it appears that the power of the crown-glass lens is of the same sign with the power of the compound, and that of the flint-glass of the opposite sign. Hence, if the compound lens is convex, as it necessarily is when used as the object-glass of a telescope, the crown-glass lens must be convex, and the flint-glass concave.

[illegible]. For the triple achromatic lens we have the equations

$$\varpi'\varphi' + \varpi''\varphi'' + \varpi'''\varphi''' = 0,$$

$$\varphi' + \varphi'' + \varphi''' = \Phi.$$

In this case, if the power of the compound lens be supposed given, there are three arbitrary quantities, namely, the powers of the component lenses, and but two equations. We are at liberty, therefore, to assume the value of one of these quantities arbitrarily; and the other two will be then determined by the equations just given.

In the usual construction of the triple achromatic lens, the middle lens is of flint-glass, and is a double concave of equal curvatures; the extreme lenses are both of crown-glass, and are equi-convex lenses of equal powers. In this case, therefore, $\varpi''' = \varpi'$, and $\varphi''' = \varphi'$. Wherefore, making $\varpi' = p.\varpi''$, the preceding equations become

$$2p\varphi' + \varphi'' = 0, \quad 2\varphi' + \varphi'' = \Phi.$$

Whence it is obvious that the two convex lenses are equivalent to a single lens of the same material and of double the power.

The triple achromatic lens, however, is susceptible of advantages (as far as achromatism alone is concerned) of which the double lens is not capable. This will be seen hereafter, when we shall enter more fully into the conditions of achromatism.

(262.) In order that the foci of the extreme red and violet rays should be united in the pencil refracted by any combination of lenses, disposed in any manner along the same axis, or the combination itself achromatic, we must have $\Delta\beta^{(n)} = 0$ (262.); or, putting for $\Delta\beta^{(n)}$ its value,

$$\pi\phi + \pi'\phi'\left(\frac{\beta}{\alpha'}\right)^2 + \pi''\phi''\left(\frac{\beta\beta'}{\alpha'\alpha''}\right)^2 + \&c. = 0,$$

in which the quantities β, α', β', α'', &c. are determined by the equations already given (262.).

When the lenses are in contact, the fractions $\frac{\beta}{\alpha'}$, $\frac{\beta'}{\alpha''}$, $\frac{\beta''}{\alpha'''}$, &c. are each equal to unit; and the equation is reduced to that of (263.).

In the case of two lenses, this equation becomes

$$\pi\phi + \pi'\phi'\left(\frac{\beta}{\alpha'}\right)^2 = 0,$$

in which the quantities β and α' are determined by the equations

$$\frac{1}{\alpha'} = \frac{1}{\beta} + \frac{1}{\theta}, \quad \beta = \alpha + \phi.$$

Now, from these equations we have

$$\frac{\beta}{\alpha'} = 1 + \frac{\beta}{\theta} = 1 + \frac{\alpha+\phi}{\theta}.$$

Wherefore, substituting this in the preceding equation, dividing the result by $\pi'\phi'$, and denoting the ratio $\frac{\pi}{\pi'}$ by p, as before, the condition of achromatism of two lenses separated by any interval is

$$p\frac{\phi}{\phi'} + \left(1 + \frac{\alpha + \phi}{\theta}\right)^2 = 0.$$

Since this result contains α, the vergency of the incident pencil, it follows that the combination which is achromatic for any one distance of the object, ceases to be so when that distance is varied; so that if two lenses separated by any interval be achromatic for parallel rays, they will not be so for near objects.

When the incident rays are parallel, $\alpha = 0$; and if we substitute for ϕ, ϕ', and θ, their values $\frac{1}{f}$, $\frac{1}{f'}$, and $\frac{1}{\delta}$, f and f' being the focal lengths of the two lenses, and δ the interval between them, the preceding equation becomes

$$p\frac{f'}{f} + \left(1 + \frac{\delta}{f}\right)^2 = 0.$$

This equation may be satisfied either by means of the focal length of one of the lenses, or the interval between them. To satisfy it in the latter way, we must solve the equation for δ, and we find

$$\delta = f\left(\sqrt{-p.\frac{f'}{f}} - 1\right).$$

Now this value will be always real, provided f and f' be of opposite signs, or one of the lenses convex and the other concave. From this, therefore, we derive the important conclusion that any two lenses of this kind, whatever be their refractive and dispersive powers, may be rendered achromatic, simply by separating them to the distance determined by the preceding formula.

When the quantity under the radical sign is equal to unity, or $\frac{f}{f'} = -p$; the value of δ becomes nothing, and the lenses must be placed in contact, agreeably to that which has been already shown (264.). When the quantity under the radical sign is less than unity, or $\frac{f}{f'} > -p$; in order that the value of δ should be positive, f must be negative, or the first of the two lenses convex. And, on the other hand, f must be positive, or the first of the two lenses concave; when $\frac{f}{f'} < -p$, or the quantity under the radical sign greater than unity.

The preceding equation suggests also a very simple method of determining the ratio of the dispersive powers of two media formed into lenses, of which one is convex and the other concave. For we have only to separate the lenses until the combination becomes achromatic. This is determined in ordinary cases by the practical optician, by applying the compound lens to form an image of a well defined white circle upon a black ground; if, on examining this image with an eye-glass of high power, the edges are observed to be free from colour, the lens is achromatic. This adjustment being made, and the interval between the lenses then measured, the ratio of the dispersive powers will be

$$p = -\frac{f}{f'}\left(1 + \frac{\delta}{f}\right)^2.$$

(267.) In forming an achromatic combination of two media, the condition which has been assumed (264.) is to unite the foci of the extreme rays in the emergent pencil. Now, if the media which are employed for this purpose act proportionally upon the rays of all colours, *i. e.* if the spaces occupied by the differently coloured rays in the spectrum are proportional to the whole spaces of dispersion, it is evident that, in uniting the foci of the extreme rays, the foci of all the intermediate rays are also united, and the combination is therefore perfectly achromatic.

It appears, however, from what has been stated (249.) that this condition does not hold, and that, in general, one of the two media acts more upon the rays of intermediate refrangibility, in proportion to the extremes, than the other; so that, though the foci of the extreme rays should be united in the emergent pencil, the focus of the intermediate or green rays will not coincide with that of the other two, and thus a dispersion will arise exactly analogous to the secondary spectrum already mentioned.

These *secondary dispersions*, arising from the *irrationality* of the coloured spaces of the spectrum, though considerably less than the first or primary dispersions, are yet too considerable to be overlooked either by the theoretical or practical optician. Dr. Blair was the first to draw the attention of scientific men to their consideration, as also to show in what manner they might be corrected*. It naturally occurred to him that as the primary dispersions were corrected by combining two media in which those dispersions were different for the same refraction; so also the secondary dispersions might be corrected by means of two such binary combinations, in which the secondary dispersions were unequal for the same refraction. The theory of such corrections, it will readily appear, is the same as that of the former; so that, when two binary achromatic combinations are employed to correct the secondary dispersions, one of these combinations must be convex and the other concave, and their focal lengths must be to one another in the ratio of their secondary dispersive powers.

The limited variety of solid transparent substances which could be used for this purpose led him to employ combinations of fluids, whose refractive and dispersive powers were known or could be easily ascertained. It is obvious that such a combination may be simplified, by taking one medium common to the two binary combinations, and the surfaces, which bound it externally, plane; for thus the two combinations may be placed together, and the bounding surfaces dispensed with; and in this manner a *triple* achromatic lens may be constructed, in which the dispersions both of the first and second order are corrected.

In the same manner as a secondary dispersion has been found to arise when the primary dispersion is destroyed by the combination of two media; so also, when three media are employed to correct the dispersions of the first and second order, it is evident that a *tertiary dispersion* will arise from the want of union of the other rays of the spectrum: and, if this be corrected by the addition of another lens, dispersions of succeeding orders will in like manner arise indefinitely. These dispersions decrease rapidly, so that in practice it is unnecessary to attend to any beyond the second. But as the theory of their correction is given by the same analysis, we shall, in what follows, inquire generally the conditions to be fulfilled by any combination of lenses, in order that it may be *perfectly achromatic.*

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* His researches connected with this subject are of exceeding interest. The account of them will be found in the Transactions of the Royal Society of Edinburgh for the year 1791.

(268.) It has been proposed by Mr. Herschel to take water, at the temperature of its maximum of density, as a standard of comparison in optical as well as in other physical inquiries, and to determine any ray of homogeneous light by its index of refraction from a vacuum into water. This being known, it is evident that the colour, and all other physical properties of the ray dependent upon its degree of refrangibility, are determined.

Accordingly, if x denote the refractive index of any ray for water, and μ that of the same ray for any other medium, μ must be a function of x, whose form will depend upon the nature of the refracting medium. We have therefore

$$\mu = \mathrm{F}(x);$$

and, if $\mu + \delta\mu$ and $x + \delta x$ be any other corresponding values of μ and x,

$$\mu + \delta\mu = \mathrm{F}(x + \delta x).$$

But the second member of this equation may be expanded into a series of the form

$$\mathrm{F}(x) + \mathrm{A}\delta x + \mathrm{B}(\delta x)^2 + \mathrm{C}(\delta x)^3 + \&\text{c.}$$

in which, A, B, C, &c. are functions of x, derived from the primitive function $\mathrm{F}(x)$, and independent of the increment δx. Wherefore, if we substitute this series for the second member of the preceding equation, and subtract the primitive equation $\mu = \mathrm{F}(x)$, there is

$$\delta\mu = \mathrm{A}\delta x + \mathrm{B}(\delta x)^2 + \mathrm{C}(\delta x)^3 + \&\text{c.}$$

Now, if in this result $x = x_{\prime}$, its least value, δx and $\delta\mu$ become, respectively, the differences between any corresponding values of x and μ and their least values; and the coefficients, A, B, C, &c. are constant quantities. Accordingly $\delta\mu$, the difference between any value of μ and its least value, is represented by a series of the ascending powers of δx, the corresponding difference of the refractive index of the ray by water, with constant coefficients.

If a, b, c, &c. be another series of constant coefficients connected with the former by the relations

$$
\begin{gathered}
(x_{,} - 1)\,\mathrm{A} = (\mu_{,} - 1)a, \\
(x_{,} - 1)^2\mathrm{B} = (\mu_{,} - 1)b, \\
(x_{,} - 1)^3\mathrm{C} = (\mu_{,} - 1)c, \\
\text{\&c. \&c.}
\end{gathered}
$$

in which $x_{,}$ and $\mu_{,}$ denote the least values of x and μ, respectively; the preceding equation assumes the form

$$\frac{\delta\mu}{\mu_{,} - 1} = a\frac{\delta x}{x_{,} - 1} + b\left(\frac{\delta x}{x_{,} - 1}\right)^2 + c\left(\frac{\delta x}{x_{,} - 1}\right)^3 + \text{\&c.}$$

(269.) In order to determine the coefficients a, b, c, &c., for each different medium, it is necessary that we should know as many corresponding values of the quantities $\frac{\delta\mu}{\mu_{,} - 1}$ and $\frac{\delta x}{x_{,} - 1}$, as there are coefficients required. The accurate observations of Fraunhofer on the refractive indices of the rays corresponding to the *fixed lines* in the spectrum furnish us with the data necessary for this inquiry. The following table is an abridgment of that given by him in his *Essay on the determination of the refractive and dispersive powers*, &c. and contains the values of the refractive indices, for different media, of the deficient rays of solar light, which correspond to seven of the principal fixed lines which he has selected as standards of comparison, and designated by the letters B, C, D, E, F, G, H. Of these B and C are in the red portion of the spectrum, the former being near the extremity; D is in the orange, E in the green, F in the blue, G in the indigo, and H in the violet.

As the refractive indices of the substances here considered all lie within the limits 1 and 2, to avoid the repetition of the integer unit, we shall subjoin the values of $\mu - 1$, instead of those of μ itself.

Refracting Medium.	*Specific Gravity.*	$\mu - 1$						
		B	C	D	E	F	G	H
Water . .	1.000	.3309	.3317	.3336	.3358	.3378	.3413	.3442
Solution of potash .	1.416	.3996	.4005	.4028	.4056	.4081	.4126	.4164
Oil of turpentine	0.885	.4705	.4715	.4744	.4783	.4817	.4882	.4939
Crown glass (1)	2.535	.5243	.5253	.5280	.5314	.5343	.5399	.5447
Crown glass (2)	2.535	.5258	.5268	.5296	.5330	.5360	.5416	.5466
Crown glass (3)	2.756	.5548	.5559	.5591	.5631	.5667	.5735	.5795
Flint glass (1)	3.512	.6020	.6038	.6085	.6145	.6200	.6308	.6404
Flint glass (2)	3.695	.6236	.6255	.6306	.6373	.6435	.6554	.6661
Flint glass (3)	3.724	.6266	.6284	.6337	.6405	.6468	.6588	.6697
Flint glass (4)	3.723	.6277	.6297	.6350	.6420	.6483	.6603	.6711

Now, in the general value of $\frac{\delta\mu}{\mu_{\prime} - 1}$ given above, we may, without introducing any error appreciable in practice, neglect all the terms of the series beyond the second as inconsiderable, the powers of the quantity $\frac{\delta x}{x_{\prime} - 1}$ decreasing rapidly. We shall accordingly apply the preceding table to determine the coefficients a and b in the expression

$$\frac{\delta\mu}{\mu_{\prime} - 1} = a\frac{\delta x}{x_{\prime} - 1} + b\left(\frac{\delta x}{x_{\prime} - 1}\right)^2.$$

For this determination it is necessary to obtain *two* corresponding values of the quantities $\frac{\delta\mu}{\mu_{\prime} - 1}$ and $\frac{\delta x}{x_{\prime} - 1}$ from the preceding table; and, as the differences $\delta\mu$ and δx should be taken from rays as wide asunder in the spectrum as possible, we will take the values of these differences which result from comparing the refractive index of the ray B, with those of the rays E and H, respectively; the refractive index of the ray B corresponding to the least values of μ and x, which we have denoted by $\mu_{\prime}$ and $x_{\prime}$. In this manner we find the two values of $\frac{\delta x}{x_{\prime} - 1}$ to be .014808 and .040193, respectively; and, if the corresponding values of $\frac{\delta \mu}{\mu_{\prime} - 1}$ be denoted by M and M$'$, there is

$$\text{M} = .014808\,a + .000219\,b,$$

$$\text{M}' = .040193\,a + .001615\,b.$$

From these equations the values of a and b, for each of the substances in the foregoing table, are found; and in this manner the following table is constructed:

Refracting Medium.	*a*	*b*
Water	+ 1.0000	+ 0.0000
Solution of potash .	0.9963	1.1326
Oil of turpentine . .	1.0615	4.5864
Crown glass (1) . .	0.8737	2.4920
Crown glass (2) . .	0.8842	2.3491
Crown glass (3) . .	0.9013	3.4900
Flint glass (1) . .	1.2901	7.6305
Flint glass (2) . .	1.3703	8.4409
Flint glass (3) . .	1.3758	8.6690
Flint glass (4) . .	1.4258	7.5770

(270.) To apply the preceding to the investigation of the conditions of achromatism of any combination of lenses in contact, let ϕ', ϕ'', ϕ''', &c. denote the powers of the component lenses, and ϕ that of the system; then, in order that the combination should be perfectly achromatic, the variation of ϕ, in proceeding from the least refrangible rays to those of *any other* species, must be nothing; that is (261.) we must have

$$\frac{\delta\mu'}{\mu'-1}\phi' + \frac{\delta\mu''}{\mu''-1}\phi'' + \frac{\delta\mu'''}{\mu'''-1}\phi''' + \&c. = 0,$$

whatever be the increments $\delta\mu'$, $\delta\mu''$, $\delta\mu'''$, &c. Wherefore, if we substitute for $\frac{\delta\mu'}{\mu'-1}$, $\frac{\delta\mu''}{\mu''-1}$, &c. their values expressed in series of the powers of $\frac{\delta x}{x-1}$, namely,

$$\frac{\delta\mu'}{\mu'-1} = a'\frac{\delta x}{x-1} + b'\left(\frac{\delta x}{x-1}\right)^2 + c'\left(\frac{\delta x}{x-1}\right)^3 + \&c.$$

$$\frac{\delta\mu''}{\mu''-1} = a''\frac{\delta x}{x-1} + b''\left(\frac{\delta x}{x-1}\right)^2 + c''\left(\frac{\delta x}{x-1}\right)^3 + \&c.;$$

the result must be nothing, whatever be the magnitude of the variable $\frac{\delta x}{x-1}$. Hence the coefficients of each of the powers of this variable must be separately equal to nothing: that is, the following equations must be satisfied:

$$a'\phi' + a''\phi'' + a'''\phi''' + \&c. = 0,$$

$$b'\phi' + b''\phi'' + b'''\phi''' + \&c. = 0,$$

$$c'\phi' + c''\phi'' + c'''\phi''' + \&c. = 0,$$

&c. &c.

On the first of these equations depends the destruction of the dispersion of the 1st order; on the second, that of the 2d order, and so forth.

The number of these equations being indefinite, it becomes impossible to satisfy them all, or, in other words, to obtain *perfect achromatism*, by the combination of any finite number of lenses of given materials. This, however, is of little importance in practice, inasmuch as the dispersions whose destruction depends upon their fulfilment form a rapidly decreasing series; so that it becomes unnecessary in practice to attend to any but the first two equations, which furnish the conditions of the destruction of the dispersions of the first and second order.

(271.) It is easy to see that the number of these equations which can be satisfied at once is one less than the number of lenses employed. Thus, with two lenses we can only satisfy the first, which thus becomes

$$a'\phi' + a''\phi'' = 0, \quad \text{or} \quad \frac{\phi''}{\phi'} = -\frac{a'}{a''};$$

by which the ratio of the powers of the lenses is determined, and therefore the power of each, if that of the compound be known. This result is identical with that obtained in Art. (264.), in which we sought only the correction of the dispersion of the first order.

With three lenses, however, we may satisfy the first two equations, which thus become

$$a'\phi' + a''\phi'' + a'''\phi''' = 0,$$
$$b'\phi' + b''\phi'' + b'''\phi''' = 0.$$

From these equations the ratios of the powers of the lenses are determined as follows:

$$\frac{\phi''}{\phi'} = -\frac{a'b''' - a'''b'}{a''b''' - a'''b''}, \quad \frac{\phi'''}{\phi'} = -\frac{a'b'' - a''b'}{a'''b'' - a''b'''},$$

which, combined with the equation

$$\phi' + \phi'' + \phi''' = \phi,$$

determine completely the powers of the component lenses when the dispersions of the first and second order are destroyed.

(272.) If any of the fractions $\frac{a'}{a''}$, $\frac{a'}{a'''}$, or $\frac{a''}{a'''}$, happens to be equal to the corresponding fraction $\frac{b'}{b''}$, $\frac{b'}{b'''}$, or $\frac{b''}{b'''}$, the numerator or denominator of one of the preceding expressions will vanish; and, consequently, the power of one of the lenses must be nothing. It is easy to see, in fact, that in this case the other two lenses satisfy the two conditions of achromatism. Thus, if $\frac{a'}{a''} = \frac{b'}{b''}$, it is evident that the equations

$$a'\phi' + a''\phi'' = 0,$$
$$b'\phi' + b''\phi'' = 0,$$

become identical, and therefore that the lenses which satisfy the first will also satisfy the second; so that the dispersions, both of the first and second order, are in this case corrected by two lenses only.

The same observation may be extended to the dispersions of succeeding orders; and it will readily appear that if two substances be so constituted that

$$\frac{a'}{a''} = \frac{b'}{b''} = \frac{c'}{c''} = \&c.$$

the lenses composed of them, whose powers are such as to satisfy the first of the conditions of achromatism,

$$\begin{gathered} a'\phi' + a''\phi'' = 0, \\ b'\phi' + b''\phi'' = 0, \\ c'\phi' + c''\phi'' = 0, \\ \&c. \ \&c. \end{gathered}$$

will satisfy all the succeeding. The condition involved in the preceding relations is evidently that of *proportionate dispersion*; and it will be easily understood, *a priori*, that if all the rays of which the solar light is composed are dispersed proportionally by two media whose total dispersions are different; by combining them so as to unite the extreme rays, the intermediate rays will be likewise united.

It is, then, a practical problem of the utmost importance, to obtain a medium which, with a different total dispersive power, shall disperse the different rays in the same proportion with some known medium, crown glass for instance. With fluid media this may be easily accomplished: for the law, according to which a medium acts upon the different rays, depends evidently on its chemical composition; and, in a fluid medium, this may be altered indefinitely until at length we obtain one of the required character.

In the course of the experiments into which Dr. Blair entered on this subject, it was found that the addition of a metal to any fluid, while it increases the whole refractive and dispersive powers, has also the effect of increasing the dispersion of the more refrangible rays in a higher proportion than that of the less refrangible. Muriatic acid, on the other hand, was found to have the opposite effect, namely, of diminishing the proportion which the dispersion of the more refrangible rays bears to that of the less. He concluded, therefore, that by combining muriatic acid with metallic solutions in due proportions a compound might be obtained, in which the law of dispersion should be exactly the same as that of crown glass, though their total dispersions were different. His anticipations were fully verified; and by enclosing the fluid thus obtained between two lenses of crown glass, he was enabled to form a compound lens of considerable aperture in which no trace of chromatic dispersion could be discovered.

It is much to be desired that investigations of this nature were extended to solid media. Could a perfectly transparent glass be made, of considerable size and uniform density, possessing the property of the fluid which Dr. Blair describes, there would be nothing to limit the perfection to which the telescope might be brought.

CHAPTER IV.

OF THE COLOURS OF NATURAL BODIES.

I.

Of the unequal Reflexion of Light by natural Bodies; and of the Colours thence arising.

(273.) We have hitherto spoken of the reflexions and refractions which light undergoes when it encounters an even surface of any form, such as the surfaces of liquids when at rest, and the artificial surfaces given to glasses or metals by grinding and polishing. In such cases we have seen the rays of light are reflected or refracted regularly in certain directions, so as to meet the eye only when placed in those directions. Such, however, is not the case with the surfaces of most natural bodies: these, on account of their inequalities, present every inclination of surface to the incident light, and, consequently, reflect or refract it in every possible direction.

To explain this more fully; if a beam of light be incident upon a plane mirror, it is evident from what has been said, that it will be reflected in a certain direction, making an equal angle with the surface of the mirror as the incident beam. Now, suppose this mirror subdivided into any number of portions, ten for instance, each of which is inclined to the incident beam at a different angle, then it is equally evident that the light, instead of being reflected in one particular direction, will be reflected in ten different directions; and if each of these portions be again subdivided, and the inclination of its parts varied, the directions in which the incident rays are reflected will be multiplied accordingly; so that if the magnitude of the partial surfaces be diminished indefinitely, and their number and position indefinitely increased, the light will be reflected in every possible direction. Now this is, in general, the case with the inartificial surfaces of natural bodies, any portion of which, however small, may be regarded as a polyhædron of an indefinite number of sides; and accordingly the light incident upon every such portion will be reflected by it in every possible direction. The reflected light, therefore, radiates from every portion of such a body, as the direct light from a self-luminous body, in all directions; and is visible to the eye any how placed.

This then forms the ground of the distinction between real bodies, whether they be self-luminous or shine by reflected light, and optical images formed by the convergence or divergence of reflected or refracted rays. The light emanating from the former in all directions, they are visible to the eye any how placed with respect to them; while that of the latter proceeding only in certain lines, they are only visible to the eye when placed in those directions.

(274.) There is no body in nature which reflects the *whole* of the light incident on it; a part of this light, in all cases, enters the body, and is either transmitted, or meeting with the parts of its substance is stifled or absorbed. Mercury, which is among the most reflective of all known substances, reflects three-fourths of the incident light, the remaining fourth being absorbed within its substance. The brightness of natural bodies depends upon the proportion which the light reflected by them bears to the incident light.

But further, not only do bodies differ from one another in the whole quantities of solar light which they reflect; but also, in one and the same body, the proportion of the reflected to the incident light is, in general, different for each species of simple light of which the solar light is composed, some bodies reflecting one species of rays more than the rest. Hence it is that natural bodies appear coloured; the colours which they exhibit being those of the predominant rays in the light reflected by them; and, accordingly, the colours of natural bodies arise from nothing else than their aptitude to reflect this or that species of homogeneous light more copiously than others. Thus minium reflects the least refrangible, or red rays, more copiously than the rest, and thence derives its colour; violets, on the contrary, reflect the most refrangible, or violet rays, more than the rest; and so of other bodies.

That such is a just explanation of the phenomena of the colours of natural bodies appears fully from the experiments of Newton,' to which we have already alluded. All bodies, whatever be their colour when exposed to solar light, exhibit the colour of the homogeneous light in which they are placed; appearing brighter, however, and more luminous in the light of their own colour, than in any other. Thus, *cinnabar* appears red when placed in a homogeneous red light, green in a green light, and blue in a blue; its brightness, however, being greatest in the red light, less in the green, and least of all in the blue. *Ultramarine*, on the contrary, appears brightest when placed in a homogeneous blue light; that brightness diminishes when it is brought into the green light, and is least of all when exposed to the red; the colour which it exhibits being in every case that of the light in which it is placed. And so of all other coloured bodies, which are always found to be most luminous in the light of their own colour.

Again, if two such bodies, as the cinnabar and ultramarine, be compared together in different lights, it will be found, that when placed in a homogeneous red light, they both appear red, the cinnabar, however, being of a brilliant red, the ultramarine of a very obscure one; but when transferred to the blue extremity of the spectrum, the order of their brightness is reversed, the cinnabar exhibiting an obscure blue, and the ultramarine a brilliant one *. From all which it is evident that the cinnabar reflects the red rays more copiously than those of any other colour, and the ultramarine the blue; and so of other bodies.

But further, this fact that each species of natural bodies reflects the differently coloured rays in different proportions, affords not only an easy explanation of the phenomena of the colours of such bodies, but is, moreover, the only ground on which they can be accounted for. For, since it appears from the experiments just mentioned, that the colours of the different species of homogeneous light are not in any respect altered by reflexion at the surfaces of natural bodies, it follows that such bodies cannot appear coloured otherwise than by reflecting the rays of that particular colour which they exhibit in daylight, or such rays as compound it by their mixture.

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* It is to be observed, that white bodies reflect the light of any particular colour more copiously even than the bodies of that colour, all coloured bodies absorbing a considerable portion of the incident light.

(275.) Let S denote the whole number of rays of whatever kind which enter the composition of a solar beam; A, B, C, D, E, F, G, the number of red, orange, yellow, green, blue, indigo, and violet rays, respectively, in the same beam; so that

$$A + B + C + D + E + F + G = S.$$

Then, if α denote the proportion of red rays which are reflected to those that are incident, $A\alpha$ will be the number of red rays in the reflected beam. In like manner the number of orange, yellow, green, blue, indigo, and violet rays in the reflected beam will be $B\beta$, $C\gamma$, $D\delta$, $E\varepsilon$, $F\zeta$, $G\eta$, respectively; β, γ, δ, ε, ζ, and η, denoting the proportions of the reflected to the incident rays for each colour. Accordingly, the whole number of rays in the reflected beam, or its intensity, will be represented by the formula

$$A\alpha + B\beta + C\gamma + D\delta + E\varepsilon + F\zeta + G\eta,$$

and the colour of the reflected beam will depend upon the relative magnitude of the coefficients α, β, γ, &c. which coefficients depend upon the nature of the reflecting substance.

Thus, in those bodies which appear of a white, grey, or any neutral colour, such as those which the clouds exhibit, the coefficients α, β, γ, &c. are all equal; such bodies reflecting all the different species of light in the same proportion. So that the preceding formula becomes

$$\alpha(A + B + C, \&c.) = \alpha S;$$

and the intensity of the shade depends upon the magnitude of the coefficient α. All such shades differ from one another in degree only, not in kind; the darkest grey differing from the most brilliant white only in the quantity of light. This was fully evinced by Newton, by comparing such bodies together under different degrees of illumination.

For bodies, which exhibit a homogeneous light of any colour, all the coefficients vanish except that of the particular colour. Thus, for bodies of a homogeneous red, all the coefficients vanish, except α, and the colour reflected by the body is represented by $A\alpha$; for those of a homogeneous blue, it is $E\epsilon$; and so for others. Such bodies, however, are not to be met with in nature; amongst those which approach nearest to them may be reckoned blood, gamboge, and ultramarine. Bodies of the most vivid prismatic colours, such as minium and vermilion, generally reflect a considerable portion of each of the different kinds of light, with a predominant portion of the light of their own colour.

When α, β, and γ are large with respect to the other coefficients; *i. e.* when the red, orange, and yellow predominate in the reflected light, we have the various shades of scarlet, orange, and the darker browns. In the composition of the latter colours, scarcely any other species of light enters but the three just mentioned, and of these the coefficients are generally small. Accordingly, when a painter wishes to form the darker shades of brown, he mixes black with red or yellow, or both, his object being to absorb the light of other colours. When the coefficients of the other colours increase, the resulting shades are such as would arise from a dilution of those mentioned with white, and we obtain the lighter yellowish tints, together with all the lighter browns.

When γ, δ, and ϵ are larger with respect to the rest; *i. e.* when the yellow, green, and blue predominate, we obtain all the various shades of green and olive. And it is remarkable that the heterogeneous green formed by the mixture of blue and yellow is of the most perfect kind, not at all distinguishable from the prismatic green without the aid of the prism.

When the larger coefficients are those of the red and blue, the combination furnishes all the beautiful shades of crimson, purple, lilac, pink, &c.

II.

Of the unequal Transmission of Light by natural Bodies, and of the Colours thence arising.

(276.) It has been already mentioned that a considerable part of the light incident upon bodies in all cases enters their substance, and is either transmitted, or meeting with the solid parts of the substance is absorbed and lost. When a body transmits freely the *whole* of the light which enters its substance, it is said to be perfectly *transparent;* and, on the other hand, it is denominated *opaque* when none of the light which enters it is transmitted, but the whole stifled or absorbed.

There is, however, no body in nature either perfectly transparent, or perfectly opaque. The most transparent of all bodies, such as air, water, glass, &c. stop a portion of the light which enters their substance; and this portion increases with the extent of the medium through which it passes, so that when the latter is sufficiently great, the portion of light transmitted becomes too inconsiderable to affect the sight. To this want of perfect transparency in the air it is owing that the brightness of all objects decreases with the distance, so that they cease altogether to be visible when that distance is considerable. Objects under water become invisible at smaller and not very considerable distances; and to the same cause is, probably, to be ascribed the fact noticed by Captain Kater, and which he has accounted for on a different supposition; namely, that objects are seen more distinctly through the Galilean telescope than in the common astronomical of the same power; the thickness of the eye-glass at its central parts being much smaller in the former case than in the latter.

On the other hand, there is no body, however seemingly opaque, which will not transmit some portion of light if reduced sufficiently in thickness. One of the densest of all known substances, gold, if beaten thin, will transmit a portion of the incident light, appearing of a greenish hue when placed between the eye and the light.

But further: the same substance does not transmit all the different species of light with the same facility. Thus, if a red liquid, contained in a vessel of a conical form, be placed between the eye and the light, at the lower part where it is thinnest it will appear of a pale yellow colour; higher up, where the thickness is somewhat greater, this yellow becomes a full yellow or gold colour; then it is changed into an orange; next into a bright red; and lastly into a dark but full red, which becomes darker as the thickness is greater, until at length it vanishes altogether, and the liquid becomes to all sense opaque. This is easily explained: the liquid transmits the red rays most easily, the yellow next in order, and so on in the inverse order of their refrangibility. Accordingly, when the thickness of the medium is small, none of the rays are intercepted, except the extreme violet and part of the indigo; and the rest, which are transmitted, compound a colour not differing much from white light, but having a shade of yellow. As the thickness is increased, the indigo and blue rays are intercepted, and the yellow of the transmitted beam becomes more full and rich. When the green and yellow are successively intercepted by the increasing thickness, the colour is gradually changed into a bright orange, then to a brilliant red, and last of all to a deep prismatic red, which is the last to suffer extinction.

(277.) The solar light being separated in the manner we have described, part being reflected, part transmitted, and the remainder stifled or lost, it would seem to follow that the colours which bodies exhibit by reflexion and transmission should be, in general, different; and, if no portion of the incident light were absorbed by the medium, the colours of the reflected and transmitted lights would necessarily be complemental, *i. e.* such as together compound whiteness. Now this is observed to be nearly the case in very many instances. Thus, Dr. Halley observed, in the course of some experiments made at a considerable depth below the surface of the sea in a diving-bell, that the back of his hand, which was illuminated by the direct light of the sun transmitted through a small glass window in the top of the bell, exhibited a brilliant rose colour; while the lower part of his hand, which was illuminated by the light reflected from the lower parts of the water, appeared green. Sea-water, therefore, transmits the red or least refrangible rays most easily, and reflects the most refrangible. Again: gold appears yellow by reflected light, while it exhibits a bluish colour inclined to green by transmitted light, when beaten sufficiently thin. Thus, also, the infusion of lignum nephriticum is of a red or yellow colour by transmitted light, while by reflected light it is blue.

There are many bodies, however, that exhibit the same colour both by reflexion and transmission. This Newton accounted for by supposing the light of that colour to be reflected by the remoter surface of the body, or by the air beyond it. In such circumstances, it is evident, the reflected light shall have suffered transmission through double the thickness of the medium, and must therefore be of the colour most readily transmitted. It is probable that there is in general a reflexion from both surfaces, and that the colour of the reflected light is that which results from the mixture. In this manner, it is evident, the colour of the reflected light may have every variety of shade, from the colour of the transmitted light itself to the colour most opposed to it. This account of the phenomenon is confirmed by the fact that the reflected colour depends, in almost all cases, on the thickness of the substance, becoming fuller and richer as that thickness is increased up to a certain limit; as may easily be observed in precious stones, and other transparent coloured substances.

(278.) We have already seen (21.), that when homogeneous light is propagated in parallel rays in a medium of imperfect transparency, the portion transmitted through any thickness of the medium, θ, will be $\text{A}\alpha^{\theta}$; in which A denotes the whole quantity of the light at its entrance into the medium, and α the ratio which the transmitted part bears to the whole after passing through a unit of thickness. It now appears that the different species of homogeneous light are not transmitted with the same facility in any known medium; and therefore that the quantity α in the preceding expression, which may be termed the *index of transmission*, is different for each different species of simple light. Accordingly, all the phenomena of colours, exhibited by transmitted light in natural bodies, will be explained by assigning a different index of transmission to each of the different species of homogeneous light of which solar light is composed.

Let A, A', A'', &c., therefore, denote the number of rays of each species in the incident light; and α, α', α'', &c. their indices of transmission; the intensity of the transmitted light, or the number of rays transmitted through any thickness of the medium, θ, will be

$$A\alpha^\theta + A'\alpha'^\theta + A''\alpha''^\theta + \text{\&c.} = \Sigma(A\alpha^\theta);$$

the intensity of the incident light being

$$A + A' + A'' + \text{\&c.} = \Sigma(A).$$

The colour of the transmitted light will depend upon the relations which the coefficients α^θ, α'^θ, &c. in this expression bear to one another; and, as these are continually varying with every variation in θ, it follows that this colour is continually changing with every change in the thickness of the medium.

If in any medium the indices of transmission were the same for each of the different species of simple light of which solar light is composed, the transmitted light would be colourless. For in this case the preceding expression becomes $\alpha^\theta\Sigma(A)$, and the colour of the transmitted light will be white, varying only in intensity as the thickness of the medium is varied. No known media, however, possess this property.

If the thickness of the medium be indefinitely small, however, the transmitted light will be colourless, and that whatever be the nature of the medium. For, if $\theta = 0$, $\alpha^\theta = 1$, whatever be the value of α; and the intensity of the transmitted light is represented by $\Sigma(A)$, which is that of the incident light. It is for this reason that the foam of all coloured liquids is colourless.

(279.) The *ultimate* colour of the transmitted beam will be evidently that of the light whose index of transmission is greatest; the power of that quantity, α^θ, being the last to become indefinitely small as the thickness is increased.

When the value of α decreases *regularly* from one point in the spectrum, the transmitted beam will exhibit all varieties of tint intermediate between absolute whiteness and the homogeneous light of that colour to which the maximum value of α corresponds. Thus, in media of this kind whose ultimate tint is *red*, the value of α decreases regularly from the red to the violet extremity of the spectrum; consequently the violet rays are the first absorbed, next the indigo and blue, and so on in order; so that the colour of the transmitted light varies through all the shades of pale yellow, full yellow, orange, bright red, and deep red, as the thickness is increased; and the last ray transmitted is the prismatic red. Of this kind are all red, orange, brown, and yellow glasses, port-wine, infusion of saffron, &c. In media of this kind, whose ultimate tint is *blue*, the value of the transmissive index decreases regularly from the most refracted to the least refracted extremity of the spectrum. Such are the blue solutions of copper. In *green* media of this nature, the value of α decreases regularly from the central rays of the spectrum to the two extremes, so that both red and violet rays are easily absorbed, the yellow and blue less so, &c. Such are green glasses, green solutions of copper, &c.

(280.) When the value of a has *two maxima*, the light transmitted through a sufficient thickness of the medium will be found, when examined with a prism, to consist principally of the two species of rays to which these maxima correspond. Such media, therefore, may be termed *dichromatic*, and the colour of the transmitted beam will be one compounded principally of the two colours above-mentioned, in proportions varying with the thickness, until finally it becomes that of the ray to which the *greatest* value of α belongs.

The changes of colour exhibited by such media are often very remarkable. To understand them it may be observed that, if the maxima values of α be at all considerable with respect to the rest, all the other terms in the value of $\Sigma(\mathrm{A}\alpha^{\theta})$ may be neglected in comparison of the two which involve these quantities, for any moderate thickness of the medium; and accordingly if α and α' denote these maxima (the former being the greater), and A and A' the number of rays of the two species in the intromitted beam, the intensity of the transmitted light will be represented by the formula

$$\mathrm{A}\alpha^{\theta} + \mathrm{A}'\alpha'^{\theta}.$$

Now, if A' be much greater than A, the second term will be, at first, much greater than the first, and therefore its colour the predominating one in the colour of the transmitted light; but α' being less than α, α'^{θ} will become indefinitely smaller than α^{θ} as the thickness increases. The proportion of the first term to the second, therefore, increases indefinitely, and the colour of the rays to which it belongs becomes the ultimate tint of the transmitted light.

There are many green media of this nature, having two maxima values of α, of which the lesser belongs to the green rays, and the greater to the red. Such media, therefore, pass from a green through an intermediate livid hue to a red. All mixtures of red and green liquids possess this property: it belongs also to sea water, as is evident from what has been already mentioned (277.). All purple media are necessarily dichromatic, their colour being compounded principally of the extreme red and violet rays; and the ultimate tint of such media is either red or violet.

There are other media, for which the index of transmission has a greater number of maxima values, and in which, therefore, the variation of colour follows a law of greater complexity. Blue media are often of this kind; and the common smalt-blue glass, of frequent use in the arts, is a remarkable instance of it. If the light transmitted through a piece of this glass of about $\frac{1}{20}$ of an inch in thickness be examined by a prism, it will be found to have four maxima of transmission, the first and greatest of which corresponds to the extreme red ray, the next to the red of mean refrangibility, the third to the mean yellow, and the last to the extreme violet. When the thickness of the glass is small, the compound colour of the transmitted beam is a pure blue: as that thickness is increased, however, none but the extreme red and violet rays are suffered to pass, and the blue colour is changed into a purple. As the thickness is still further increased, the violet rays are gradually extinguished, the purple assumes more and more of a reddish hue, until finally it becomes a deep red corresponding to the extreme red of the spectrum.

(281.) If several media be combined together, the light transmitted through all is that which is left after the action of each; and it will be readily seen that it is the same in whatever order these media are disposed. When two media are combined, therefore, whose transmitted lights contain no common homogeneous ray, no ray whatever can be transmitted through both. Thus, if one medium transmits the homogeneous red rays, and another the green or blue only, all the rest being extinguished, no ray whatever can pass through both, and the compound is perfectly opaque; although each of the substances, taken separately, was transparent. This phenomenon was observed by Hooke before any thing was known of the true nature of colours.

By combining two or more media, we are enabled also to insulate a homogeneous ray in a state of purity scarcely attainable by any other means. Thus, if a full red glass be employed along with the smalt-blue glass already mentioned, the combination is impermeable to every ray except the extreme red ray of the spectrum. In this manner we are enabled to identify this ray under different circumstances, and to compare it with itself when acted on by different media; and are thus furnished with a definite standard of the utmost value in optical researches.

(282.) When a beam of solar light, after dispersion by a prism, is transmitted through a coloured medium of sufficient thickness having several maxima of transmission, the phenomena which present themselves are very remarkable. The regular gradation of colour, which the solar image would otherwise apparently exhibit, is altogether destroyed, and the spectrum appears to consist of several detached portions of coloured light separated from one another by dark bands, or intervals, precisely analogous to Fraunhofer's *fixed lines;* the rays belonging to these points in the spectrum being completely absorbed. Thus, when a thin piece of the smalt-blue glass, already mentioned, is employed *, the red light is separated into two well-defined portions, parted from one another by a broad and perfectly black band, and wholly undistinguishable in colour. Of these the lowest or least refrangible corresponds to the extreme red of the spectrum, and is a perfectly homogeneous light; the other, which corresponds to the meanly refrangible red ray, is nearly homogeneous, and without the slightest shade of orange. The orange is altogether obliterated, the next colour being a well-defined band of pure and full yellow, which is separated from the second red by a small well-defined black line, and from the green by a dark interval. The green is dull, but the colours increase in fulness and purity to the extremity of the spectrum, the extreme violet suffering very little loss.

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* Encyclopædia Metropolitana, article Light.

From this experiment we learn that rays which affect the sense with precisely the same sensation of colour, as the two reds already mentioned, differ widely in refrangibility, being separated by a broad and black interval; while, on the other hand, rays strongly contrasted in colour have at their adjacent extremities nearly the same degree of refrangibility; the second red and the yellow being separated by a very narrow line, and suffering no mixture whatever where they approach nearest. Does it not appear to follow from this that the connexion between the colour of a ray and its degree of refrangibility is not so complete as was supposed by Newton? and that the analysis of solar light by refraction is not the only analysis of which it is capable? True it is that each ray in the solar spectrum has its particular shade of colour and peculiar degree of refrangibility, which cannot be altered by reflexions or refractions, however numerous; but may not each such ray be, nevertheless, compound, and consist of several rays, *different in colour*, but having the *same degree of refrangibility?* If such were the case, it is evident that no refractions could ever separate them, and we must have recourse to some other property, such as this difference in their transmissibility through coloured media, as affording the means of disuniting them.

(283.) These hypotheses seem to receive a strong confirmation from the facts we have mentioned; and Dr. Brewster accordingly supposed the solar light to be composed of three primary colours, *red*, *yellow*, and *blue*, each of which has every degree of refrangibility within the known limits; so that the solar spectrum consists, according to this hypothesis, of three spectra of different colours overlapping one another, and having each its maximum of intensity at the point where the rays of that colour are most intense in the solar spectrum. Thus, according to this supposition, each ray of the solar spectrum is compounded of three others in varying proportions, the red predominating in the red rays of the spectrum, the red and yellow in the orange, the yellow in the rays of that colour, the yellow and blue in the green, the blue in the blue, and the red and blue in the violet.

The hypothesis of three primary colours was first advocated by Mayer; but seems to have had, at that time, no other confirmation than that derived from the facility with which the prismatic colours may be imitated by the combination of the three already mentioned. Independently, however, of the absence of all proof, thus afforded, of the *actual* composition of the colours of the spectrum, the distinction of primary colours, thence arising, seems, in a great measure, arbitrary, inasmuch as any particular colour may be compounded of different others in various ways. Accordingly we find that Dr. Young has assumed the *red*, *green*, and *violet* as the primary colours, and shown that all others may be compounded of them in different proportions. And from an examination of the spectrum, which first exhibited the fixed lines, Dr. Wollaston concluded that the primary colours were *four* in number, namely, *red*, *green*, *blue*, and *violet;* and that of 100 parts, into which the whole length of the spectrum was supposed to be divided, those occupied by these several colours were as the numbers 16, 23, 36, and 25, respectively.

PART III.

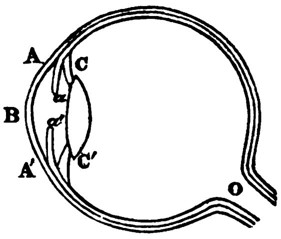
OF VISION.

CHAPTER I.

OF THE EYE, AND OF UNAIDED VISION.

(284.) THE adjoining figure represents a horizontal section of the human eye; the form of which is nearly spherical, the fore part, ABA′, however, being more convex than the rest.

The several humours of which the eye consists are all contained in a thick tough coat, the more convex part of which, ABA′, called the *cornea*, is transparent, and of a consistent horny character. The remainder of this exterior coat is called the *sclerotica:* it is opaque and white, and forms what is in common language designated as the *white of the eye.*



The interior of the sclerotica is lined by a second and thinner coat of a softer substance, called the *choroid membrane,* which is firmly attached to the sclerotica by a circular band extending round the edge of the cornea, and called the *ciliary ligament.*

At the junction of the choroid with the sclerotica, and supposed to be a continuation of the former membrane, arises the *uvea,* an opaque membrane or screen, having an aperture in the centre, ad', called the *pupil*, through which the rays of light incident on the eye are admitted.

The uvea consists of muscular fibres, by the contraction or expansion of which the aperture of the pupil is contracted or enlarged. The use of this is to moderate the quantity of light incident upon the sensitive part of the eye. In very strong lights the pupil is contracted, in weak ones it is enlarged; its aperture varying, in the eyes of adult persons, from about $\frac{1}{10}$th to $\frac{1}{4}$th of an inch. This contraction and dilatation of the pupil is involuntary, and arises on the excess or defect of the light itself. It is much greater in some of the lower animals than in man. The pupil in the eye of the cat, for instance, is almost closed in daylight. In the human eye the pupil is always circular: its form varies in the eyes of other animals. In the feline tribe, as the dog, the cat, &c. the vertical diameter is invariable, and the form of the pupil varies from a circle to a vertical right line. In ruminating animals, on the contrary, the greater and invariable diameter is the horizontal one.

The anterior surface of the uvea is differently coloured in different persons, varying through all the shades of green, blue, brown, and gray. This part is sometimes called the *iris*, and its colour determines that of the eye. The posterior surface of the uvea is covered with a black mucus, which is evidently intended to absorb any light which may happen to fall upon it, and thus to prevent internal reflexions which would disturb the vision.

The interior of the choroid membrane is covered with a very black mucous substance, called the *pigmentum nigrum*[*], in which is imbedded the *retina*, the third and innermost coat of the eye. This is a network of extremely fine nerves, branching from the optic nerve, o, which proceeds directly from the brain, and enters obliquely at the back of the eye, and at the inner side towards the nose.

As the retina communicates directly with the brain itself, so the other coats of the eye, the sclerotica and the choroid, are supposed by some to be continuations of the *dura mater* and the *pia mater*, the outer and inner coats of the brain. This point, however, does not seem to be at all established.

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[*] The use of the pigmentum nigrum is, like that of the mucus which clothes the interior of the uvea, to absorb the light which enters the eye when it has excited the retina, and thus to prevent internal reflexions.

(285.) Within the eye, and a little behind the uvea, is suspended a soft, transparent, jelly-like substance, cc′, called the *crystalline lens*, of the form of a double-convex lens of unequal radii, the anterior surface being less curved than the posterior. The structure of this substance is fibrous, being composed of laminæ or layers successively superimposed, as may be seen in the lens of a boiled fish's eye; and each coat consists of a vast number of fibres diverging from two poles, the line joining which coincides with the axis of the eye. The crystalline is contained in a thin transparent capsule; and kept in its place by the *ciliary processes*, a projecting fold of the choroid membrane, which arises at the same place as the uvea, and is a little convex towards it. The anterior surface of this muscle, like the posterior surface of the uvea, is covered with a black mucus to absorb erratic rays.

The space before the crystalline humour, and between it and the cornea, is filled with a transparent fluid resembling water, and thence denominated the *aqueous humour*. The space behind the crystalline, and between it and the retina, is filled with another transparent fluid somewhat more viscous than the former, and called the *vitreous humour*. These two humours are, like the crystalline, contained in transparent membranous capsules of extreme delicacy and tenuity.

The aqueous and vitreous humours do not differ sensibly from water in specific gravity. The aqueous humour, indeed, consists principally of pure water; containing, besides, small quantities of albumen and gelatine, together with muriate of soda. The latter ingredients, however, together do not exceed eight per cent. The vitreous humour is said not to differ sensibly from the aqueous in chemical composition. The crystalline is somewhat heavier than water; and contains a much larger portion of albumen and gelatine than the other two humours.

In their refractive powers the aqueous and vitreous humours differ very little from that of water. The refractive index of the aqueous humour is 1.337, and that of the vitreous humour 1.339; that of water being 1.336. The refractive power of the crystalline is greater, its mean refractive index being 1.384. The density of the crystalline, however, is not uniform; but increases gradually from the outside to the centre. According to Dr. Brewster and Dr. Gordon, the refractive indices of the outer coat, the middle, and the central parts, are 1.3767, 1.3786, and 1.3999, respectively. This increase of density serves to correct the aberration, by increasing the convergence of the central rays more than that of the extreme parts of the pencil.

(286.) The observations of M. Petit respecting the dimensions of the parts of the human eye are the most detailed of any we possess at present*. From these, compared with the modern measurements of Wollaston, Young, Brewster, &c. we conclude that the axis of the human eye, measured from the outer surface of the cornea to the retina, is about .95 of an inch; and that the portions of it occupied by the cornea and the different humours are as follows: cornea .04 of an inch; aqueous humour .11; crystalline .17; vitreous humour .63. Hence the portion occupied by the vitreous humour is about two-thirds of the whole length of the axis.

These proportions are very different, however, in other animals; in most of which the portion occupied by the crystalline humour bears a greater proportion to the whole than in man. In the eyes of fishes, in particular, the crystalline humour is by far the largest of the three, and in the herring's eye M. Cuvier found that the portion of the axis occupied by the crystalline was five times that occupied by each of the other humours.

The other dimensions of the eye, derived from the same sources, are as follows:

	Inches.
Interior transverse diameter of the eye, .	.90
Chord of the cornea (vertical), . .	.46
Chord of the cornea (horizontal), . .	.49

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* These are contained in the Memoirs of the Royal Academy of Sciences of Paris for the Year 1730.

	Inches.
Chord of the crystalline,	.37
Radius of external surface of cornea,	.33
Radius of anterior surface of crystalline,	.33
Radius of posterior surface of crystalline,	.24

It appears from these results that the circumference of the cornea is not circular, its vertical and horizontal diameters being as 15 to 16 nearly.

It will be also observed, in conformity with what has been already stated (285.), that the anterior surface of the crystalline is less curved than the posterior, the radii of these surfaces being to one another in the ratio of 4 to 3 nearly. This ratio, however, is very different in different animals. In the eye of the ox, for instance, these radii are as 3½ to 1; and in some animals the curvature of the anterior surface is greater than that of the posterior. In the eyes of fishes the form of the crystalline is spherical, and the increase of density towards the centre very considerable. The reason of this appears to be that, as the other humours of the eye are nearly of the same refractive power as the medium in which they dwell, the destruction of aberration, as well as the refraction itself, must be almost wholly the work of the crystalline.

It must be observed, finally, with respect to the curvatures of the bounding surfaces, that the greatest diversity exists both in different individuals, and even in the same individual at different periods of life; the surfaces becoming uniformly flatter with age. The preceding results exhibit their average values for persons in the middle time of life.

(287.) The bounding surfaces of the refracting media, however, are not spherical, as has been generally supposed, but *spheroidical*. This remarkable fact was long since suspected by M. Petit, but of late has been placed in the clearest evidence by the accurate measurements of M. Chossat. This author has found that the cornea of the eye of the ox is an *ellipsoid* of revolution round the *greater* axis, this axis being inclined *inwards* about 10°. The ratio of the major axis to the distance between the foci in the generating ellipse he found to be 1.3; and this agreeing very nearly with 1.337, the index of refraction of the aqueous humour, it follows that parallel rays will be refracted to a focus, by the surface of this humour, with mathematical accuracy (207.).

The same author found likewise that the two surfaces of the crystalline lens are *ellipsoids* of revolution round the *lesser* axis; and it is somewhat remarkable that the axes of these surfaces do not coincide in direction either with each other, or with the axis of the cornea, these axes being both inclined *outwards*, and containing with each other, in the horizontal section in which they lie, an angle of about 5°.

It is not to be supposed, however, that the curvatures of the surfaces of the eye are the same in degree, or even in kind, in other animals. The same author has found that the cornea in the eye of the elephant is an *hyperboloid.* For the details of his experiments we must refer to the original memoir*.

(288.) Having thus far explained the structure of the eye, we shall, in the next place, proceed to consider it as the instrument of vision.

The eye, we have seen, consists of three refracting media, of which the two extremes have very nearly the same refractive power as water, the intermediate one a refractive power somewhat greater. Accordingly the light incident from any object on the eye will undergo a refraction at each of the bounding surfaces of these media, and this refraction in each case tends to give convergence to the incident rays. Thus the first refraction takes place at the convex surface of the aqueous humour, which, being a denser medium than the surrounding air, will give a convergence to the refracted pencil. The extreme rays of this pencil are then intercepted by the uvea; and the central part of the pencil, being transmitted through the pupil, is incident upon the convex surface of the crystalline, which, being denser than the aqueous humour, will increase the convergence; and, finally, it falls upon the concave surface of the vitreous humour, which, being rarer than the medium from which the pencil has emerged, also adds to the convergence, and thus completes the refraction. By means of these successive refractions each pencil of incident rays is brought to a focus at or near the retina, and thus an image is formed there corresponding in form and colour with the object from which the rays flow.

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* *Sur la courbure des milieux refringens de l'œil chez le bœuf.*—Annales de Chimie, tom. x.

These images may be readily exhibited by taking off the outer coats from the back of the eye of a newly killed animal: miniature pictures of external objects will be seen depicted there as upon a screen of roughened glass.

It is evident that the axes of the several pencils, which go to form the images, must intersect before they reach the retina. Hence the image on the retina will be inverted with respect to the object. The point in which these axes intersect is called the *focal centre* of the eye. Its position, it is obvious, will vary both with the distance of the object and its magnitude; but it is never far distant from the posterior surface of the crystalline*.

(289.) The *apparent magnitude* of any object is measured by the angle contained by the axes of the extreme pencils; that is, by the angle subtended by the object at the focal centre. It is therefore proportional to the linear magnitude of the object divided by its distance from the centre of the eye.

There has been much discussion amongst writers on the eye with respect to the magnitude of the *minimum visibile*, or the apparent magnitude of the least visible object. It is usually stated that a single object upon a ground of an opposite colour, as a black circle upon a white ground, or a white circle upon a black ground, cannot be seen by the generality of eyes under a smaller angle than one minute. Most persons, however, can perceive as distinct two points subtending an angle of this magnitude; and some are found who can distinguish two such points under an angle of $20''$ only. It is even stated that a single object, if sufficiently bright, and placed at the proper distance for distinct vision, may be seen under an angle of two or three seconds.

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* Authors differ much respecting the place of the focal centre of the eye; some placing it in the centre of the pupil, others in the centre of the crystalline, and others, lastly, in the centre of the eye. The position assigned to it above seems to be that given by calculation.

It appears to us that there is no limit whatever to the angle under which an object is visible, and that the power of exciting the sensation of vision will depend, not on the magnitude of the visual angle, but on the quantity of light proceeding from the object in relation to that proceeding from surrounding objects.

This conclusion seems to be fully confirmed by the results of Harris's experiments. Of these the principal are as follows: 1. That a detached object may be seen under a smaller angle than the parts of a compound object of the same magnitude. 2. That the least angle under which any object may be seen will depend upon its brightness as compared with that of surrounding objects. Thus a white square upon a black field was seen under a smaller angle than a square of the same dimensions upon a field shaded lightly with Indian ink. 3. That the least angle under which an object may be seen will depend upon its other dimensions. Thus if different lines, of the same breadth, be drawn upon the same ground, and viewed at different distances, the longer lines will be visible at greater distances than the shorter, and their breadth subtend therefore a smaller angle. And, further, the same author concludes, from a comparison of the observations of objects of different forms, that the *areas* of the least visible objects are the same, all other circumstances being alike.

These facts seem to leave little doubt as to the conclusion that the power which any object possesses of exciting the retina depends solely on its relative quantity of light; and if this be admitted, it follows that the angle subtended by the least visible object may be indefinitely diminished, if its brightness be proportionally increased. Accordingly we find that the fixed stars produce a distinct and vivid impression on the retina, although the angle they subtend at the eye is less than any we are able to estimate by the most accurate observations.

This conclusion is perfectly compatible with the supposition that the image on the retina must have some definite magnitude in order to produce an impression. We have strong reasons for believing that the eye is not free from chromatic dispersion;

and from this it follows, even when the light is collected to a focus exactly at the retina, that the image of a point will be diffused over a small circle there of a definite magnitude. And, as the density of the light in this circle diminishes from the centre to the circumference, the extent of it which is capable of impressing the retina will depend upon the whole quantity of light emanating from the luminous point. This will explain the reason why the brighter fixed stars appear larger than those which are fainter, although the angle subtended by none is of any definite magnitude.

(290). The greatest extent of field which the eye is capable of receiving at once will be measured by the angle which the diameter of the pupil subtends at the focal centre of the eye, the retina being supposed sensible to the impressions of light within these limits. This extent of field, however, is found to be different in different directions. Dr. Young found that, when the visual axis was fixed in a certain direction, the angle of vision of his own eye extended upwards 50°, downwards 70°, inwards 60°, and outwards 90°. These internal limits of the field of view correspond nearly with the external limits formed by the projecting parts of the face, when the eye is directed forwards and somewhat downwards, which is its natural position. This difference of the extent of the field in different directions may arise from a difference in the extent of the sensible portion of the retina: it seems, however, to be fully accounted for by the fact that the focal centre of the eye is not symmetrically situated with respect to the pupil. It was ascertained by Dr. Young, in the course of some observations made for the purpose of determining the position of the insensible spot on the retina, that the extremity of the visual axis of the eye, or the line joining the centre of the pupil with the focal centre, is distant the $\frac{1}{20}$th of an inch from the point of the retina immediately opposite to the pupil.

We have already noticed some of the adaptations of this wonderful piece of mechanism which are in strict accordance with the results of the most refined science: such is the elliptical figure of the cornea, and the connexion between that figure and the refractive index of the humour which it contains; such too is the increase of density of the crystalline towards the centre; both tending to correct the aberrations which would be otherwise produced by these media. We have another such subject of admiration in the *position* of the *uvea*, the use of which, we have seen, is to intercept the extreme rays, and thus also to correct the aberration. Had it been placed in front of the eye, or at any considerable distance from its present position, it must have greatly limited the field of view, which is measured by the angle which the aperture in the uvea, or pupil, subtends at the focal centre of the eye; and if, to increase the field of view, this aperture had been enlarged, it would not have served its present purpose of intercepting the extreme rays. But, in its present position, which is not far distant from the focal centre, it not only intercepts the extreme rays of each pencil, but also admits pencils whose axes are very widely divergent from one another, and therefore opens a wide field of view to the eye.

As the uvea itself suggested in all probability the use of the *diaphragm*, or eye-stop, in telescopes, so this peculiar propriety of its position has been advantageously imitated by Dr. Wollaston in the construction of his *periscopic* lenses. These consist of two plano-convex lenses, united together at their plane sides, between which is interposed a diaphragm, having an aperture in the centre. The field of view is obviously much greater than in lenses whose aperture is equal to that of the diaphragm.

It must not be supposed, however, that the vision is distinct throughout the whole of the wide field of view which the eye possesses. Whether it arises from the want of perfect sensibility in the parts of the retina remote from the visual axis, or from the distortion of oblique pencils, or from both these causes conjointly, it is found that those objects alone are seen distinctly which are in the centre of the field; the remaining parts of the field serving merely to convey notices to the mind of the *presence* of objects situated there. The field of *perfect vision* does not extend more than 5° from the visual axis in every direction. Comparetti even supposes that distinct vision is confined to a single point of the retina at the extremity of the visual axis; and he conceives that objects of sensible magnitude are discerned by a rapid and imperceptible motion of the optic axis over their several parts, the impression produced by each remaining a certain length of time, so that they are all combined into a single sensation.

This limitation of the field of distinct vision, however, is attended with no practical disadvantages of any moment: for, by the revolution of the eye in its orbit, the axis of the eye has a range of 55° in every direction; and by this means the field of perfect vision has an actual extent of 110°, independently of any motion of the head or of the rest of the body.

(291.) It is a curious fact that there exists a small portion, within the limits of the field of view of the eye, which is invisible; or, in other words, there is a part of the back of the eye which is insensible to the impression of the rays of light, and which for that reason has been called the *punctum cæcum*. It is that spot at which the optic nerve is introduced, and at which that nerve is not yet subdivided into those delicate and sensitive fibres of which the retina is composed. This may be observed by placing two patches of white paper upon a dark wall, the line joining them being horizontal, and about the height of the eye from the ground. If then one eye be closed, and the other directed to one of the patches (the one to the left hand if the right eye be used, and *v. v.*), the other, to which the eye is not directed, becomes invisible on retiring from the wall to about four or five times the distance of the patches from one another, and, the distance being further increased, becomes again visible. The experiment is rendered more remarkable by placing a third patch beyond this in the same right line, which will continue visible when the middle one disappears.

It is evident that the quotients arising from the division of the interval between the patches by the distances of the eye from the wall when one of them first disappears, and reappears on retiring further, are the tangents of the angular distances of the farther and nearer edges of the spot, respectively, from the extremity of the optic axis. These angles being in this manner ascertained by observation, it is found that the angular distance of the centre of the insensible spot from the extremity of the optic axis is about 14°; and that the angle subtended by the diameter of this space at the centre of the eye is about 5°. These angles being known, the linear magnitude and position of this space are determined. Dr. Young found that, in his own eye, the distance of the centre of the optic nerve from the extremity of the optic axis was nearly $\frac{1}{6}$th of an inch; and that the diameter of the insensible part was about $\frac{1}{30}$th of an inch.

(292.) The *apparent brightness* of any luminous object is proportional to the density of the light in its image on the retina, which is as the quantity of light directly, and inversely as the space over which it is diffused. Now, if the aperture of the pupil be supposed invariable, and no light be lost in its passage through the air, the quantity of light incident on the pupil, and therefore that which falls on the retina, varies inversely as the square of the distance of the object (13.); and it is evident (289.) that the area of the image on the retina, or the space of diffusion, varies in the same ratio. Hence, on these suppositions, the brightness of the luminous object is invariable at all distances from the eye.

In actual experience, however, we find that the brightness of all luminous objects diminishes with the distance, and that at considerable distances they become altogether invisible. The reason of this is, that the air is not a perfectly transparent medium, but on the contrary stops a considerable portion of the light which is transmitted through it. The law of this diminution has been already explained (23.).

(293.) We have seen that the eye is endowed with peculiar adaptations, whose function seems to be to correct the errors arising from the form of the refracting surfaces; and experience proves that the eye, if not perfectly aplanatic, is at least sufficiently so for all the purposes of perfect vision.

It occurs then naturally to inquire whether there is any thing in the arrangements of this organ, or in the perceptions of vision themselves, which would lead us to conclude that the eye is also free from chromatic error. It seems now to be the received opinion amongst physiologists that the eye is not achromatic. If a lucid point be viewed through a prism its image will be a coloured spectrum, which, if the eye were achromatic, should appear extended into a *line* of light. This, however, is not the case: when the eye is directed to one extremity of the spectrum, the other extremity is dilated into a sensible breadth, and the form of the coloured image becomes triangular; proving plainly that the different rays of the spectrum are not brought to a focus at the same distance within the eye. By a comparison of the dimensions of this triangular space, Dr. Wollaston suggested that the dispersive power of the eye might be determined; and Dr. Young concluded, from some experiments established on these principles, that the mean dispersive power of the eye is about one-third of that of crown-glass.

Experience shows that the want of perfect achromatism in the eye forms no material impediment to the perfection of vision. And it has been even contended by some that the image of each point of an object, formed on the retina, must have a certain determinate magnitude in order to impress the nerves of that organ; and therefore that the theoretical perfection of the eye, considered as an optical instrument, would be incompatible with its performance as a material organ.

(294.) There is, however, a much more important limitation to the powers of the eye arising from another cause. If the form of the eye were *invariable*, it is evident that it would collect such rays only to a focus on the retina which had a *certain degree* of divergence at their incidence on the eye; that rays incident with a less degree of divergence, or proceeding from remoter objects, would be brought to a focus before they reached the retina; while those which proceeded from a nearer distance would, after refraction by the humours of the eye, tend to a focus beyond it; and that in either case the image on the retina itself would be confused, the rays proceeding from each point of the object being spread over a small circular space of sensible diameter.

Now every hour's experience proves that the eye is capable of seeing distinctly objects whose distances vary within very wide limits. To every eye there is a certain distance at which it is able to see distinctly without any effort: this is called the *distance of perfect indolent vision*. Beyond this distance no internal effort whatever will enable the eye to see distinctly; but within that distance it is capable of discerning objects with perfect distinctness up to a certain limit, which is called the *least distance of distinct vision*. The least distance of distinct vision varies in common eyes from about six to eight inches.

The greatest distance, or the distance of perfect indolent vision, varies within much wider limits, and cannot be so readily ascertained. It is however so considerable, and the divergence of rays proceeding thence so small, that it is usually considered that most eyes are fitted to bring *parallel* rays to a focus on the retina. All optical instruments are accordingly adapted in such a manner that the rays shall be parallel at their incidence on the eye.

(295.) The question then arises, by what changes of conformation in the eye does it thus become adapted to near distances? Is it by a change in the position of the crystalline lens? or by a change of the curvature of its surfaces, or of that of the cornea? or, lastly, is it produced by a change in the configuration of the entire eye, and an elongation of its axis? Each of these different opinions has been advocated, and much discussion has arisen as to the true one. We shall briefly consider the arguments in favour of each.

With respect to the first opinion—that, namely, which ascribes the effect to a change of place of the crystalline—little need be said. The eye being spherical and full of an incompressible fluid, such a motion, it is evident, could not be effected without a corresponding extension of the cornea; an effect which the strength and thickness of that coat, compared with that of the muscles by which such a motion is to be effected, render in the highest degree improbable.

The most plausible of these opinions is that which combines a change of curvature of the cornea with a change in the figure of the entire eye; an opinion which has been advocated by Ramsden and Sir Everard Home. According to these authors, the *recti* or straight muscles, which move the eye in its orbit, by their simultaneous contraction produce a lateral pressure upon the eye, which both elongates it in the direction of its axis, and, through the medium of the fluids within, forces out the cornea and renders it more convex. To demonstrate the existence of the latter effect, Ramsden fixed the head of a spectator in such a manner that it could not move; and, having placed a microscope so as to examine the eye in profile, he made the wire of the microscope coincide with the edge of the cornea while the eye of the spectator was directed to a distant object. The eye of the spectator being then directed to some near object, the cornea appeared immediately to project beyond the wire *.

This experiment, however, is not so decisive as it appears at first view; as the slightest change in the direction of the axis of the eye would produce a change in the appearance of the cornea as viewed in profile, and make it seem to advance or retire. Indeed the experiment recorded by Dr. Young, in his valuable Essay on the Mechanism of the Eye †, seems to leave little doubt that the power of adaptation of the eye to different distances does not reside in the cornea. The socket of a small lens belonging to a microscope being nearly filled with water, the eye was immersed in it so as to be in contact with the water throughout the whole of the fore part of its surface, and the escape of the fluid prevented by covering the edge of the socket, where it was attached to the eye, with wax. The cornea being thus in contact with the water, and the fluids on either side of it of the same refractive power very nearly, it is evident that any change in the curvature of this coat could, under these circumstances, produce no change whatever in the refraction. Notwithstanding this, Dr. Young found that his eye possessed a power of adaptation to different distances within the same limits as before.

The same author concluded, from some observations made upon his own eye when confined in the direction of its axis, that the power of accommodation to different distances does not depend in any manner upon an elongation of the axis of the eye.

According to Dr. Young, the accommodating power resides in the crystalline, which he supposes to be susceptible of a change of figure, becoming more convex when the eye is di-

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* In the first Memoir published by Sir Everard Home on this subject, he and Mr. Ramsden attributed the entire effect to this supposed change of curvature of the cornea. Subsequent experiments, however, led them to modify this opinion, and to attribute only one-third of the effect to this cause, the rest of the adaptation being supposed to be effected by the elongation of the axis.

† Philosophical Transactions for the year 1801.

rected to near objects, and again recovering its former shape when distant objects are viewed. The regular fibrous structure of the crystalline, already mentioned, seems to indicate a muscular structure, for which we can hardly suppose any office, if it be not employed in some adaptation of this nature. And the numerous experiments of Dr. Young upon persons who had been couched for the cataract seem to leave little doubt that the principal part, if not the whole, of the adaptation in question must be the work of the crystalline; persons deprived of that lens being found, in a great degree, deficient in the power of seeing distinctly at different distances.

Before we leave this subject, it may be observed, that a change in the aperture of the pupil tends, as far as it goes, to adapt the eye to different distances; the diminution of that aperture limiting the breadth of the cone of rays, which converge to a point beyond the retina, and therefore diminishing the space over which they are diffused there. This change in the aperture of the pupil, however, though it may conspire in producing the effect in question, is not sufficiently considerable to account for it altogether; and besides, in these variations, the pupil seems to be more affected by the quantity of light than by the distance of the object from the eye.

(296.) This adaptation of the eye to different distances is, however, obviously limited; and when the curvature of the surfaces, and therefore the power of the eye in its natural state, is too great or too small, it may be impossible to adapt it to different distances without artificial assistance. Thus, in *short-sighted* persons the curvature of the surfaces of the eye is too great, and all rays, except those proceeding from a near distance, are brought to a focus before they reach the retina. Such persons, therefore, can only see near objects distinctly. The defect is remedied, and the person enabled to see remote objects, by the aid of a *concave* lens, the effect of which is to increase the divergence of the incident rays, and therefore fit them for being brought to a focus on the retina *. In *long-*

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* This subject will be considered more fully in the 2d section of the following chapter.

sighted persons, on the contrary, the curvature of the surfaces of the eye is too small, and all pencils, excepting those which proceed from very distant objects, converge to a focus behind the retina. The vision of such persons is adapted to near objects by the use of a *convex* lens, which diminishes the divergence, or even gives a convergence to the incident rays, and thereby enables the eye to bring them to a focus on the retina. This defect of vision is common among aged persons, the cornea becoming flatter with age, and therefore the power of the eye diminished. It is always found, too, in those persons who have undergone the operation of couching for the cataract. In this disorder the crystalline lens becomes opaque, and the only remedy known consists in removing it, and allowing its place to be supplied by the aqueous humour: and this being of a less refractive power than the crystalline, the power of the eye is diminished.

These defects of vision are often brought on by habit. Thus, persons who are accustomed to look much at near objects, as engravers, are apt to become short-sighted; and, on the other hand, those whose attention is generally fixed on distant objects, as savages who spend the most of their time in the pursuit of game, become long-sighted, and are unable to see near objects distinctly. This is easily explained. The eye and its attendant mechanism being frequently used in one particular state, become gradually fixed in it, and the power of alteration is lost by disuse.

(297.) The eye, however, though the chief organ of vision, does not constitute the whole of the material apparatus by which it is performed. In this, as well as in the other senses, the nerves connecting the organ with the brain, and the brain itself, have each their parts to perform; and any derangement of the nerves or brain is as fatal to the sensation as a malformation of the organ itself.

In the case of the eye, the impression produced by the images on the retina is communicated to the optic nerve, and by means of that to the brain, where, by the laws of our compound being, it becomes the condition on which the sensation of vision is produced in the mind. That these several steps of the process are equally essential to vision is fully established by experience. Thus, if through want of transparency in the humours of the eye, or any other cause, the image on the retina is either imperfectly formed or wholly wanting, the vision will be in proportion defective. This is the case in the disease called the *cataract*, in which the crystalline lens loses its transparency, and therefore stops either the whole or part of the light incident upon it: the image on the retina is thus either wholly effaced, or becomes extremely indistinct, and blindness, either total or partial, ensues. Again, the humours of the eye may perform their functions perfectly, and an accurate image be depicted on the retina; and yet, through the insensibility of the optic nerve or some other cause, the impression produced on the retina may not be communicated to the brain, and thus no vision produced. This is the case in a paralysis of the optic nerve, which produces, while it lasts, total blindness.

Thus far, then, do we know with certainty of the *material* part of the process in vision. Attentive physiologists may be able to trace it farther; but, whatever be their discoveries, there is one step of the process, in this as well as in every other case of perception, which will for ever elude the utmost efforts of human sagacity; that, namely, which connects the impression made upon the body with the consequent affection of mind. Negatively, indeed, we know that the former cannot be the *cause* of the latter, although in our present state it is the indispensable condition on which, by the appointment of the Author of our being, the phenomena of mind are developed.

(298.) What has been said will enable us to give an answer to the question on which much absurd discussion has been raised; namely, why objects appear *erect*, though their images on the retina are inverted? The question, in fact, assumes that these images in themselves *constitute* vision; whereas, all that we know about them in reference to vision is simply this, that the rays diverging from the several points of the object are made to converge to corresponding points in the retina, and there excite impressions which are communicated, by means of the optic nerve, to the brain; and that the perception thence arising has a regular but inexplicable correspondence with the manner in which these rays converge on the retina, or, in other words, with the images formed there.

But further: it is now well ascertained that these perceptions of vision do not, *directly*, give us any idea of the *position* or even of the existence of external objects. It is true the several changes in the latter have corresponding perceptions annexed to them, by which we learn to judge of them; but it is by *experience* only that we come to know this regular correspondence, and learn to conclude, that what affects our sight in such and such a manner shall also affect our touch in a corresponding manner. This association is formed at so early a period of life, that all traces of its formation are quite obliterated; and we conclude that to be an original perception of sight, which is only an associated one of touch. From all this it manifestly follows that the only thing necessary to enable us to judge of the position of external objects by sight is, that there should be a steady correspondence between these positions and the impressions which the rays proceeding from them produce on the retina.

(299.) On the same principles we may solve the question, why, as there is an image formed in each eye, we do not see two objects instead of one. The explanation of this is so clearly given by Mr. Herschel [*], that we cannot do better than quote his words.

" As we have two eyes, and a separate image of every external object is formed in each, it may be asked, *why do we not see double?* and to some the question has appeared to present much difficulty. To us it appears, that we might with equal reason ask, *why*—having two hands, and five fingers on each, all endowed with equal sensibility of touch and equal aptitude to discern objects by that sense—*we do not feel decuple?* The answer is the same in both cases: it is a matter of habit. Habit alone teaches us that the sensations of sight correspond to any thing external, and to what they correspond. An object (a small globe or wafer suppose) is before us on a table; we direct our eyes to it, *i. e.* we bring its images on both retinæ to those parts which habit has ascertained to be the most sensible and best situated for seeing distinctly; and having always found that in such circumstances the object producing the sensation is one and the same, the idea of unity in the object becomes irresistibly associated with the impression. But while looking at the globe, squeeze the upper part of one eye downwards, by pressing on the eyelid with the finger, and thereby forcibly throw the image on another part of the retina of that eye, and double vision is immediately produced, two globes or two wafers being distinctly seen, which appear to recede from each other as the pressure is stronger, or approach, and finally blend into one, as it is relieved. The same effect may be produced without pressure, by directing the eyes to a point nearer to or farther from them than the wafer; the optic axes in this case being both directed away from the object seen. When the eyes are in a state of perfect rest, their axes are usually parallel, or a little diverging. In this state all near objects are seen double; but the slightest effect of attention causes their images to coalesce immediately. Those who have one eye distorted by a blow see double, till habit has taught them anew to see single, though the distortion of the optic axis subsists.

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[*] Encyclopædia Metropolitana, Part 19.

“ The case is exactly the same with the sense of touch. Lay hands on the globe, and handle it. It is *one:* nothing can be more irresistible than this conviction. Place it between the first and second fingers of the right hand in their natural position. The right side of the first and left of the second finger feel opposite convexities; but as habit has always taught us that two convexities so felt belong to one and the same spherical surface, we never hesitate or question the identity of the globe, or the unity of the sensation. Now cross the two fingers, bringing the second over the first, and place the globe on the table in the fork between them, so as to feel the left side of the globe with the right side of the second finger, and the right with the left of the first. In this state of things the impression is equally irresistible, that we have two globes in contact with the fingers; especially if the eyes be shut, and the fingers placed on it by another person. A pea is a very proper object for this experiment. The illusion is equally strong when the two forefingers of both hands are crossed, and the pea placed between them.

" Such is, undoubtedly, a sufficient explanation of single vision with two eyes; yet Dr. Wollaston has rendered it probable that a physiological cause has also some share in producing the effect, and that a semi-decussation of the optic nerves takes place immediately on their quitting the brain, half of each nerve going to each eye, the right half of each retina consisting wholly of fibres of one nerve, and the left wholly of the other, so that all images of objects out of the optic axis are perceived by one and the same nerve in both eyes, and thus a powerful sympathy and perfect unison kept up between them, independent of the mere influence of habit. Immediately in the optic axis, it is probable that the fibres of both nerves are commingled; and this may account for the greater acuteness and certainty of vision in this part of the eye."

CHAPTER II.

OF VISION THROUGH A SINGLE LENS.

I.

General Principles of Vision through a single Lens.

(300.) It is required to determine the *visual angle* under which an object is seen through a lens.

Let m and m' denote the linear magnitudes of the object and its image formed by the lens, a and a' their distances from the lens: then (195.)

$$m' = m \frac{a'}{a}.$$

But the distance of the image from the eye is equal to the sum of the distances of the image from the lens, and of the eye from the same, or $= a' + d$, d denoting the distance of the eye from the lens. Wherefore, if θ denote the visual angle,

$$\tan.\theta = \frac{m'}{a' + d} = \frac{m}{a\left(1 + \frac{d}{a'}\right)}.$$

The eye is necessarily always at the side of the lens remote *from* the incident light: when, therefore, the image is at the same side, or a' *negative* (which is always the case when there is a *real* image), there is no limit to the increase of the visual angle dependent upon the position of the eye; for the value of $\tan.\theta$, which in this case is $\frac{m}{a\left(1 - \frac{d}{a'}\right)}$, may be increased indefinitely by making d more and more nearly equal to a'. The nature of vision, however, imposes a limit, the eye not being capable of *distinct vision* when the rays diverge from a point nearer than a certain distance. If the eye be placed at this distance from the image, we have $d - a' = \lambda$, λ denoting the *least distance of distinct vision*, which is for most eyes about six inches; and the tangent of the greatest visual angle will be

$$\tan. \theta = - \frac{m}{\lambda} \frac{a'}{a}.$$

The object in this case will appear inverted, as is evident from the negative sign.

When a' is *positive*, or the image at the side of the lens *towards* the incident light, the visual angle will be greatest when $d = 0$; that is, when the eye is close to the lens. In this case the tangent of the visual angle becomes

$$\tan. \theta = \frac{m}{a};$$

from which it appears that the angle in this case is equal to that which the object subtends at the lens, as is otherwise evident. Such is the greatest value of the visual angle dependent on the position of the eye; a value which may be increased indefinitely by diminishing a, the distance of the object from the lens. For distinct vision, however, there is a limit to the diminution of a, and therefore to the increase of the visual angle. For, since the eye is at the lens, a', the distance of the image from the lens, cannot be less than λ the least distance of distinct vision. Accordingly the greatest value of $\frac{1}{a'}$ is $\frac{1}{\lambda}$, and therefore the greatest value of $\frac{1}{a}\left(= \frac{1}{a'} - \frac{1}{f}\right)$ is $\frac{1}{\lambda} - \frac{1}{f}$. Hence the greatest visual angle under which the object can be seen distinctly, in this case, is determined by the equation

$$\tan. \theta = m\left(\frac{1}{\lambda} - \frac{1}{f}\right).$$

In *concave* lenses, in which f is positive (157.), this is always less than $\frac{m}{\lambda}$, the tangent of the greatest angle under which the object can be seen distinctly by the naked eye; in *convex* lenses, f being negative, it is always greater: and in either case the ratio of these tangents, or the magnifying power, is

$$1 - \frac{\lambda}{f}.$$

(301.) In order to see, generally, in what manner the visual angle varies with the varying position of the eye and of the object, let us substitute, in the general expression of tan. θ, for $\frac{1}{a'}$ its value $\frac{1}{a} + \frac{1}{f}$, and we find

$$\tan.\theta = \frac{m}{a + d + \frac{ad}{f}}.$$

Hence, when the object coincides with the principal focus of rays proceeding in the opposite direction, $a = -f$, and therefore $\tan.\theta = -\frac{m}{f}$, a constant quantity. Again: when the eye coincides with the principal focus of rays coming in the opposite direction, $d = -f$, and therefore $\tan.\theta = -\frac{m}{f}$; the same as before. Accordingly, the object being at the principal focus, the apparent magnitude does not vary with the position of the eye; and the eye being at the principal focus, the apparent magnitude does not alter with the position of the object; the apparent magnitude in either case being equal to that under which the object would appear at a distance equal to the focal length of the lens.

When the object coincides with the lens, $a = 0$, and $\tan.\theta = \frac{m}{d}$. And when the eye coincides with the lens, $d = 0$, and $\tan.\theta = \frac{m}{a}$. When, therefore, the object or the eye is brought up to the lens, the apparent magnitude is equal to that under which the object would appear to the naked eye.

(302.) But, whatever be the magnitude and position of the object, and whatever the power of the lens through which it is viewed, the extent of the greatest visual angle will be necessarily limited by the aperture of the lens, the tangent of that angle when greatest being equal to

$$\frac{\text{A}}{d};$$

A denoting the semi-aperture of the lens. The greatest linear extent of an object, visible through a lens in any position, may be called the *field of view.* Its magnitude is at once ascertained by equating the general value of the tangent of the visual angle to this its greatest value; by which means we obtain the greatest value of m, namely,

$$m = \text{A}\,a\left(\frac{1}{d} + \frac{1}{a} + \frac{1}{f}\right).$$

The linear extent of the field of view, therefore, varies *cæt.·par.* as the aperture of the lens.

When the object is in the principal focus of rays coming in the opposite direction, $a = -f$, and $m = -\text{A}\,\frac{f}{d}$. When the eye is in the principal focus of rays proceeding in the opposite direction, $d = -f$, and $m = \text{A}$. In this latter case, therefore, the linear extent of the field is equal to the semi-aperture of the lens, whatever be the position of the object.

When the object coincides with the lens, or $a = 0$, it is evident that the preceding value of the field is reduced to $m = \text{A}$; so that the linear extent of the field, in this case, is independent of the position of the eye. On the other hand, when the eye coincides with the lens, $d = 0$, and the field is infinitely great, as is otherwise evident.

(303.) The *apparent brightness* of an object, seen through a lens, is equal to that of the object seen by the naked eye, on the supposition that no light is lost in its passage through the lens.

For the quantity of light incident upon the pupil of the eye, diverging from any point of the image, is to that incident upon the same, diverging from the corresponding point of the object, in the inverse ratio of the squares of the distances of the image and object from the eye. The quantities of light, therefore, incident upon the pupil, diverging from the entire object and image, are as their areas directly, and inversely as the squares of their distances; that is, as the squares of the angles which they subtend to the eye, their areas being as the squares of their linear magnitudes. But the spaces occupied by their images on the retina are in the same ratio; and therefore, the quantities of light in their respective images on the retina being as the spaces over which it is diffused there, the density of the light in these images, or their apparent brightness, will be the same.

There will always, however, be some light lost in its passage through the lens, which will occasion the brightness of the object seen through it to be somewhat less than that of the same object seen by the naked eye; and this loss of light will increase with the thickness of the lens, so as to produce a very sensible difference in the brightness of an object seen through a thick and through a thin lens.

In the foregoing reasoning it has been assumed that the aperture of the lens is sufficiently large, that the cones or cylinders of rays proceeding from each point of the image shall fill the pupil of the eye. When this is not the case, the quantity of light, and therefore the brightness, will be evidently diminished in the ratio of the area of the pupil to that of the section of the cone or cylinder. Now the linear section of the cone of rays diverging from any point of the image (at the place of the eye), is to the aperture of the lens, as the distance of the eye from the image to that of the lens from the same; that is, in the ratio of $d + a' : a'$. Wherefore, if A' denote the semi-diameter of this section,

$$\mathrm{A}' = \mathrm{A}\left(1 + \frac{d}{a'}\right).$$

When this quantity, therefore, is less than the semi-aperture of the pupil, the brightness of the object seen through the lens is less than that of the same object seen by the naked eye, in the duplicate ratio.

When the image is infinitely distant, or the emergent rays parallel, $A' = A$; and the brightness of the object, seen through the lens, is less than that of the same object to the naked eye, in the duplicate ratio of the aperture of the lens to that of the pupil; as long as the former is less than the latter.

II.

Of Spectacle-Glasses and Reading-Glasses.

(304.) The *distinctness* of objects seen by the naked eye depends chiefly on the accurate convergence of the rays of each pencil to as many distinct points on the retina; and the eye, we have seen (294. 5.), is furnished with an organization by means of which it can accommodate the refractions of its surfaces to the distinct vision of objects whose distances vary within very wide limits. When, however, the refracting surfaces of the eye are too much or too little curved, this power is limited on the one side or the other; the refraction of the eye in the former case being too great for the distinct vision of distant objects, the divergence of the rays proceeding from which is very small; and in the latter too small for the distinct vision of near objects, from which the divergence is considerable. These defects, it was soon seen, might be remedied by the use of a concave or convex lens, by which we are enabled to diminish or increase, as it were, the refracting power of the natural instrument; and accordingly this application of lenses was made long previous to the invention of the more complicated dioptrical combinations *.

The theory of spectacle-glasses is as simple as their construction. It is evident that the distance at which we *desire to see* distinctly by the aid of the lens, and that at which we

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* The invention of spectacle-glasses is referred to the year 1290; and they were long in use before the accidental combination of two of them led to the invention of the telescope.

actually see by the naked eye without effort, must be the conjugate focal distances of the lens before and after refraction respectively. Hence, if these distances be denoted by a and a', and the focal length of the lens by f, we have

$$\frac{1}{f}=\frac{1}{a'}-\frac{1}{a};$$

by which the focal length of the lens required is known.

For *short-sighted* persons who desire to see distinctly at a considerable distance, $\frac{1}{a}=0$, very nearly, and therefore

$$f=a'.$$

The lens employed, therefore, must be a *concave* lens, whose focal length is equal to the distance at which such persons can see distinctly with the naked eye without effort.

In the case of *long-sighted* persons, on the contrary, the distance a', at which they can see distinctly with the naked eye, is very great; so that $\frac{1}{a'}=0$, very nearly, and therefore

$$f=-a.$$

That is, the lens must be *convex*, and its focal length equal to the distance at which the person desires to see by its assistance.

(305.) The focal length of the lens being determined, by reference to the distance at which objects may be seen distinctly by the naked eye without effort, the power of distinct vision which it confers is not limited to one particular distance. For, if d and δ denote any conjugate focal distances, before and after refraction by the lens respectively, it is evident that the object will be seen distinctly through the lens at any distance, d, whose conjugate focal distance, δ, lies within the limits of distinct vision to the naked eye; and these limits of the values of δ being tolerably wide, it is evident that the values of d will have a corresponding range determined by the relation which subsists between them.

If the limiting values of δ be denoted by δ' and $\delta_{,}$, and the corresponding values of d by d' and $d_{,}$, we have

$$\frac{1}{d'}=\frac{1}{\delta'}-\frac{1}{f}, \qquad \frac{1}{d_{,}}=\frac{1}{\delta_{,}}-\frac{1}{f}.$$

When a concave lens is employed to assist a short-sighted person, f is taken equal to δ', the greatest distance of distinct vision to the naked eye; so that the limiting values of d are

$$d' = \infty, \qquad d_{\prime} = \frac{f\delta_{\prime}}{f - \delta_{\prime}}.$$

The range of distinct vision with the lens, therefore, extends from an infinite distance to the distance $\frac{f\delta_{\prime}}{f - \delta_{\prime}}$; the lesser of these limits being greater than $\delta_{\prime}$, the least distance of distinct vision to the naked eye, in the ratio of $f : f - \delta_{\prime}$.

In the case of a long-sighted person, δ', the greatest distance of distinct vision to the naked eye, is infinite; wherefore, changing the sign of f in the preceding equations, since the lens employed is convex, we have

$$d' = f, \qquad d_{\prime} = \frac{f\delta_{\prime}}{f + \delta_{\prime}}.$$

The range of distinct vision extends therefore, in this case, from a distance equal to the focal length of the lens to the distance $\frac{f\delta_{\prime}}{f + \delta_{\prime}}$; which lesser limit is less than the least distance of distinct vision to the naked eye in the ratio of $f : f + \delta_{\prime}$. The magnitude of the range is

$$d' - d_{\prime} = \frac{f^2}{f + \delta_{\prime}};$$

and increases with f. The higher, therefore, the power of the lens, or the less its focal length, the less also will be its range of distinct vision.

(306.) When an object is in the principal focus of a convex lens, the rays emerge parallel, and are therefore adapted to the vision of a long-sighted person. But, the object being brought ever so little nearer to the lens, the refracted rays will diverge, and it is evident that this divergence may be increased to any desired magnitude by approaching the object to the lens. A *convex* lens, therefore, may be employed to assist a near-sighted person, as well as one who is far-sighted, in the vision of *near objects*; and the assistance in this case is rendered by an increase of the visual angle, by which the object is exhibited enlarged or magnified.

Such is the common *reading-glass*, which is a convex lens of considerable aperture, whose focal length depends upon the degree of magnifying power required, and varies, in general, between one and two feet. The eye being brought close to the lens, the visual angle under which the object is seen is equal to that which it subtends to the naked eye (301.): the object, however, being within the limits of distinct vision, will appear magnified as compared with itself when seen distinctly by the naked eye. When the image is brought to the least distance of distinct vision, in which case the visual angle is greatest, the relative magnitudes of the object as seen by the lens, and by the naked eye at the least distance of distinct vision, will be (300.).

$$1 + \frac{\lambda}{f}, \quad . \qquad .$$

which therefore expresses the magnifying power.

(307.) In looking through spectacles when the eye is directed to any part of the field remote from the centre, the rays which are incident upon the pupil after refraction through the lenses pass through the latter with very considerable obliquity; and therefore the marginal parts of the field will be much distorted and confused. This obliquity, however, and the consequent confusion, will be in a great degree remedied by making the inner surfaces of the lenses concave; and it is obvious that, the nearer the curvature of these surfaces approaches to that of the cornea, the less will be the obliquity of the marginal pencils. Such is the simple principle of Dr. Wollaston's *periscopic spectacles*, the lenses of which are of the form of the *meniscus* or the *concavo-convex*, with their concave surface turned towards the eye; the former being intended for the long-sighted, and the latter for the short-sighted person. Where *extent of field* is required, as in the open air, this form is much to be preferred to the common one. When, however, the eye is steadily directed to the centre of the field, as in reading, or in the examination of any minute object, they are inferior to the common forms, inasmuch as the aberration of the central pencils is considerably increased.

As the eye requires a greater extent of field laterally than vertically, the shape of the lenses of spectacle-glasses is usually *oval*, instead of circular, the greater diameter being horizontal.

(308.) When the eye is immersed in water, the first and most considerable of its refractions is altogether lost. For the aqueous humour is of the same refractive power as water, very nearly (285.); and consequently, the cornea being bounded by surfaces which are nearly parallel, the rays will pass from water into the aqueous humour without undergoing any refraction. Hence, the remaining surfaces of the eye being unable to complete the refraction, the spectator is immediately placed in the condition of a far-sighted person, and will require the aid of a powerful convex lens in order to see distinctly.

The lenses employed by *divers*, if they be formed of crown-glass, and be equi-convex, must have the curvatures of both surfaces equal to that of the cornea. For, in order that the refraction of such a lens may be equal to that of the cornea, which it is intended to supply, the focal length of the lens, *in water*, must be equal to that of the cornea *in air*. Now, if μ denote the absolute refractive index of water, and μ' that of glass, the *relative refractive index* from water into glass will be $\frac{\mu'}{\mu}$ (118.); and the focal length of an equi-convex lens of glass, placed in water, will be

$$\frac{r'}{2(1-\frac{\mu'}{\mu})},$$

r' denoting the radius of either surface. But, if r denote the radius of the cornea, its focal length is

$$\frac{\mu r}{1-\mu}.$$

And equating these values, according to the condition above mentioned, we obtain

$$r' = 2\frac{\mu'-\mu}{\mu-1}r.$$

Now if we substitute for μ and μ' their values $\frac{4}{3}$ and $\frac{3}{2}$, the refractive indices of water and crown-glass respectively, we find

$$r' = r.$$

The radius of each surface of the lens, therefore, should be about the one-third of an inch; that being, very nearly, the magnitude of the radius of the cornea (286.).

III.

Of the single Microscope.

(309.) The great impediment to the vision of minute objects—namely, the smallness of the visual angle under which they are seen—might appear at first sight to be easily removeable, and to any desired extent, by simply diminishing their distance from the eye. To this diminution of distance, however, we have seen, the nature of vision imposes a limit, the eye being incapable of bringing rays to a focus on the retina whose divergence exceeds a certain magnitude. Accordingly the distance determined by this limit is denominated the *least distance of distinct vision*; and the angle, which any small object subtends to the eye at that distance, is the *greatest angle* under which it can be seen distinctly by the naked eye.

If the object be placed, however, in the focus of a convex lens, the angle under which it will appear is equal to that which it would subtend to the naked eye at a distance equal to the focal length of the lens (301.); while the rays of the several pencils, proceeding from the several points of the object, emerging parallel, will be brought to a focus on the retina without effort. Consequently the object will appear at once distinct, and magnified to the same extent as if we were enabled to approach it to a distance equal to the focal length of the lens. A lens of high power, thus employed, is called a *single microscope*.

(310.) The tangent of the visual angle under which the object is seen by the aid of such a lens is

$$\frac{m}{f};$$

m denoting the linear magnitude of the object, and f the focal length of the lens (301.). But the tangent of the angle which the object would subtend to the naked eye, at the least distance of distinct vision, is

$$\frac{m}{\lambda};$$

and the ratio of these tangents, which, the angles being small, is nearly equal to that of the angles themselves, and is therefore the measure of the magnifying power, is

$$\frac{\lambda}{f}.$$

Accordingly the magnifying power is increased by diminishing the focal length of the lens, or increasing its power. The focal lengths of lenses, fitted up as single microscopes, are usually $1\frac{1}{2}$ inch, 1 inch, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{10}$, $\frac{1}{20}$ of an inch. In examining minerals or flowers a high power is not required, and accordingly the focal lengths of the lenses which are employed in the botanical or mineralogical microscope are seldom less than $\frac{1}{4}$th of an inch. For various other purposes, however, it is desirable to obtain as high a power as possible; but, when a higher power is sought than that furnished by a lens of $\frac{1}{20}$th of an inch focal distance, it is usual to employ, instead of lenses, small spherules of glass; the construction of the latter being easier than that of the former, when the curvature is considerable.

(311.) The substitution of small glass spherules for lenses was first suggested by Dr. Hooke, who describes the method of their construction in his *Micographia*, published in the year 1665. They are attended with this disadvantage—that their distance from the object examined is necessarily much less than in a lens of equivalent power. For the distance of the principal focus of a glass sphere from its surface is equal to half the radius (165.), and is therefore one-third of its focal length measured from the centre; wherefore, neglecting the thickness of the lens, the distance of an object in its principal focus from its surface is three times as great as the corresponding distance in the sphere of equivalent power. On account of the smallness of this distance, the object is liable to be touched, and therefore in many cases to be destroyed, in adjusting the spherule to distinct vision; while, in the examination of moist objects, which require a thin plate of glass or talc to be interposed between them and the lens, in order to prevent the evaporation from interfering with the vision, the spherule becomes utterly useless when of high power.

The most powerful spherules ever made were those sent to the Royal Society by Di Torrè, of Naples. One of these is said to be the $\frac{1}{144}$th of an inch in diameter; and its focal length, therefore, is equal to the $\frac{1}{192}$ of an inch. The available power of the single microscope, however, seems to be limited by the constitution of the eye. It was observed by Huyghens, and since fully confirmed by the observations of others, that the vision will be imperfect if the diameter of the cylindrule of rays, which are incident upon the eye emerging from a lens of any kind, falls below a certain limit, which he has fixed at the $\frac{1}{72}$ of an inch. But the diameter of this cylindrule of rays is equal to the aperture of the lens or spherule: this aperture then must not be less than the $\frac{1}{72}$ of an inch, even supposing the whole of it to be effective; and therefore the focal length of a glass spherule, which is equal to three-fourths of its diameter, cannot be less than the $\frac{1}{96}$th, or, in round numbers, the $\frac{1}{100}$th of an inch.

Another disadvantage, attendant on the employment of lenses or spherules of small aperture and high power, is the want of sufficient brightness.

When an object is placed in the focus of a lens or spherule, the breadth of the cylinder of rays, proceeding from each point of the image, is equal to the aperture of the lens; and, when that aperture is less than that of the pupil of the eye, the apparent brightness of the object, seen through the lens, is less than that of the same object seen by the naked eye, in the duplicate ratio of that aperture to that of the pupil (303.). The apparent brightness, therefore, decreases as the square of the aperture; and, when the aperture is very small, becomes so inconsiderable as to require the object to be strongly illuminated.

As the aperture increases, the brightness increases in the duplicate ratio, up to a certain limit, at which it arrives when the aperture of the lens becomes equal to that of the pupil. The apparent brightness then becomes equal to that of the object seen by the naked eye, and cannot be increased by any further increase of aperture.

(312.) The *confusion* of vision, arising from spherical aberration, is measured by the area of the circle of aberration in the image on the retina. Now the diameter of this circle is proportional to the angle which the diameter of the least circle of aberration, in the image formed by the lens, subtends to the eye, or at the lens q. p.; and, if this angle be denoted by ϵ, we have (190.)

$$\epsilon = \frac{2\varrho}{\delta} = \tfrac{1}{2}\text{K}\text{A}^3;$$

in which A denotes the semi-aperture of the lens, and K the coefficient of the square of that quantity in the expression of $d\alpha'$ (173.). But, *cæt. par.*, K is proportional to the cube of the power of the lens: for, if the ratio of the curvatures of the surfaces of the lens be denoted by m, that is, if $\varrho = m\varrho'$, we have (156.)

$$\varphi = (\mu - 1)(m - 1)\varrho' = (\mu - 1)\left(1 - \frac{1}{m}\right)\varrho;$$

and these values of ϱ and ϱ' being substituted in the expression for $d\alpha'$ (173.), it is evident that the result will be of the form $d\alpha' = \text{L}\varphi^3\text{A}^2$; L being a function of μ and m, whose magnitude depends accordingly on the particular form and material of the lens. We have therefore

$$\text{K} = \text{L}\varphi^3 = \frac{\text{L}}{f^3}, \quad \text{and} \quad \epsilon = \tfrac{1}{2}\text{L}\left(\frac{\text{A}}{f}\right)^3.$$

Hence the diameter of the circle of aberration on the retina, which is measured by ϵ, is proportional, *cæt. par.*, to the cube of the quotient of the aperture divided by the focal length of the lens. And therefore, in order that lenses similar in form, and of the same material, may exhibit objects with equal distinctness, their apertures must be as their focal lengths.

When the aperture and focal length of the lens are given, the confusion arising from spherical aberration will depend upon the magnitude of the coefficient L, which we have seen depends upon the material of which the lens is composed and upon its form. Thus it appears from (172. 173.), that when the lens is a *plano-convex* of *crown-glass*, having its plane side turned towards the object, the value of L is $\frac{7}{6}$. In a *plano-convex* of the same material, but turned in the *opposite* direction, it is equal to $\frac{9}{2}$. And in the *equi-convex* lens of the same substance it is $\frac{5}{3}$. Of the three forms, therefore, the best is the plano-convex having its plane side turned towards the object; the aberration of the lens in this position being less than when turned the opposite way in the ratio of 7 to 27.

The best form, however, or that in which the aberration is a *minimum*, is the *crossed lens* with its less curved surface turned towards the object (175.). When this lens is formed of crown-glass, we have seen the radii of its surfaces are in the ratio of 1 to 6; and the value of L becomes equal to $\frac{15}{14}$, which is less than that of the plano-convex in its best position in the ratio of 45 to 49. The aberration of the latter, accordingly, exceeds that of the lens of best form by the $\frac{1}{11}$th part of the whole only; and, as it is much easier of construction, it is the usual form of lenses fitted up for the single microscope, unless where high power is required, in which case the equi-convex is that generally employed.

(313.) The aberration of the central pencil may be further diminished, or even altogether destroyed, by combining two or more lenses together of suitable forms. Thus: if two plano-convex lenses of equal focal lengths be placed in contact, with their plane sides outwards, the aberration will be $\frac{5}{7}$ths of the single equivalent plano-convex lens, and $\frac{7}{9}$ths of the aberration of the equivalent lens of best form; as will readily appear on making the proper substitutions in the formulæ of (171), (183.). Two lenses in contact, however, may be rendered completely *aplanatic*, as far as the central pencil is concerned, by a proper adjustment of the curvatures of their surfaces. The determination of the forms of two lenses, which form an aplanatic combination when placed in contact, is given in the equations of (185.); in which we have only to introduce the condition of the parallelism of the emergent pencil. The general equations are simplified when the lenses are of the same material, as is generally the case in combinations of this kind, the constants depending upon the refractive index being the same for both lenses. We leave the development of these equations to the reader; and shall merely observe that the form of one of the lenses is altogether arbitrary, there being two arbitrary quantities in the equation of condition. If the lens next the eye be assumed of the *best form*, as in the combinations proposed by Mr. Herschel, the other will be found to be a deep *meniscus* with its concave surface next the object. Such a combination has been found to perform exceedingly well when employed as the object-glass of the compound microscope.

(314.) The errors of form are of a more various and complicated nature in their effect on those pencils which pass through the lens *excentrically;* such as, in fact, are all the pencils which reach the eye proceeding from any part of the field but its centre. There will be a *distortion* of the marginal parts of the field, arising from a difference in the inclination of the emergent pencils to the axis. There will be an *indistinctness* either at the margin or at the centre of the field, resulting from the impossibility of adjusting both to such a distance from the lens, as will ensure the parallelism of the emergent pencils for all parts of the field. And, finally, from this difference in the vergency of the pencils proceeding from the centre and from the edges of the field, the field of view will appear to be *convex* instead of flat; the pencils proceeding from the central parts emerging divergent, when those which proceed from the margin of the field emerge parallel.

In lenses of very small aperture these errors are inconsiderable and may be disregarded. When the objects to be examined, however, are at all large, larger apertures are required, and the errors of which we have been speaking become of such magnitude as to demand our attention. They may be corrected, severally, by combining two or more lenses of proper forms; but as the combination, which diminishes one source of error, frequently enlarges another, it will be simpler and more effective in practice to correct all by diminishing the aperture of the lens, provided such a diminution can be rendered compatible with a sufficient extent of field. Now this object is attained in the *periscopic lenses* invented by Dr. Wollaston, which have been already noticed (290.). They are composed of two plano-convex lenses united together at their plane sides, between which a thin plate of brass is interposed, having a small circular aperture in its centre. In this construction it is evident a considerable field of view is preserved, while at the same time all the advantages of diminished aperture, in correcting the aberration both of the central and excentrical pencils, are obtained. The diameter of the aperture which Dr. Wollaston found to be most convenient was one-fifth of the focal length of the compound lens.

In order to remedy the loss of light which is produced by doubling the number of surfaces, Dr. Brewster proposed to fill the aperture with some fluid, such as oil of turpentine or Canada balsam, which is nearly of the same refractive power as the glass. The same effect, as this author has observed, may be more perfectly attained by cutting a groove in a sphere or thick double-convex lens of glass, so as to leave in the centre a circular portion equal to the proposed aperture. When a sphere is employed, either in this form or constructed of two hemispheres with an interposed diaphragm, it has this further advantage—that the rays of all the pencils, as well from the marginal parts of the field as from its centre, pass through the refracting surfaces perpendicularly, or nearly so, so that the errors of oblique incidence are almost entirely corrected.

(315.) It remains to say a few words on the employment of other transparent substances in the construction of the lenses of single microscopes.

It is evident that if a lens could be formed of any perfectly transparent substance, which combined a high refractive power with a moderate dispersive power, it would possess a great advantage over those in common use. For, the greater the refractive power of the substance, the less will be the curvatures necessary to produce a given refraction; and the less, accordingly, will be the errors of figure as well in the excentrical as in the central pencils. Now the property above mentioned is possessed in a high degree by the diamond and by most of the precious stones; and therefore these substances, if formed into lenses, might be expected to perform with an accuracy far superior to the common lenses of glass. A few years since, accordingly, Mr. Pritchard undertook, at the suggestion of Dr. Goring, who has bestowed much attention on the improvement of microscopes, to construct a lens of diamond. The difficulties which he had to overcome in working this hard substance were very considerable: he at length, however, succeeded, and the first diamond microscope was finished in the year 1826. The focal length of this lens, which was double convex, was about the $\frac{1}{30}$th of an inch.

It will be easy to estimate the extent of the advantage thus gained. When an object is placed in the focus of a plano-convex lens, having its plane surface next the object, the value of L, the coefficient of the angle of aberration, is (172, 173.)

$$\text{L} = \frac{\mu^3 - 2\mu^2 + 2}{2\mu(\mu - 1)^2}.$$

Now when the lens is of *crown-glass*, in which $\mu = \frac{3}{2}$ nearly, the value of L is $\frac{7}{6}$; while in a *diamond* lens, in which the index of refraction $\mu = \frac{5}{2}$ nearly, $\text{L} = \frac{41}{90} = \frac{4}{9}$ nearly *. So that the aberration in the latter case is less than in the former in the ratio of 8 to 21; and the confusion of vision thence arising, which is proportional to the area of the circle of aberration on the retina or to the square of the angle of aberration, will be less in the ratio of 1 to 7 nearly.

The same artist has constructed lenses of the other precious stones; but none of them seemed to be so well fitted for this purpose as the *diamond* and the *sapphire*, many of the others producing a second image by their property of double refraction. The refractive index of sapphire is about 1.8; and substituting this value for μ in the value of L, it is found to be equal to $\frac{169}{288}$, or equal to $\frac{7}{12}$ nearly. Hence the aberration of a plano-convex lens of this substance is one-half of that of the equivalent lens of crown-glass. This mineral possesses the further advantage of having a very low dispersive power, its dispersive index being = ,026 only; which is but half that of the more dispersive kinds of flint-glass. The chromatic aberration, therefore, is proportionally reduced. The facility of working this substance also, as compared with the diamond, renders its construction far less expensive.

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* The aberration may be further reduced to nearly one-half of this quantity, by making the lens of *best form;* which in this case is a *meniscus* having its concave surface next the object, the radii of its surfaces being in the ratio of 2 to 5 (175.).

(316.) A convenient microscope for temporary purposes may be readily constructed by perforating a thin plate of brass, and inserting a drop of water in the aperture. The edges of this aperture are to be made as thin as possible, in order that the fluid may assume the requisite form, which will be that of a double convex lens. An instrument of still simpler construction is suggested by Dr. Brewster: it consists merely of a drop of some transparent fluid, such as Canada balsam or turpentine varnish, upon a thin plate of glass whose surfaces are exactly parallel. A plano-convex lens will thus be formed whose curvature, and therefore its focal length, will vary with the position of the drop with respect to the plate; the curvature being greatest when the drop is vertically beneath the plate. The former of these substances will soon harden, and, if preserved from dust, continue perfect for a considerable time. Dr. Brewster has formed lenses in this manner, which he has employed with advantage both as the object-lenses and eye-lenses of a compound microscope.

CHAPTER III.

OF VISION THROUGH ANY COMBINATION OF LENSES.

I.

General Principles of Vision through any Combination of Lenses.

(317.) In tracing the progress of light through any system of lenses, two things are to be attended to. 1st, The course of a pencil of rays diverging from any one point of the object. The several foci of this pencil determine the places of the several *images* of the object. 2dly, The course of the *axes* of these several pencils. These axes, all passing through the centre of the object-glass, or first lens of the system, may be considered as constituting a pencil of rays flowing from that centre; and, consequently, an image of the object-glass will be formed at its several foci. The extreme rays of this pencil of axes determine the *field of view*, the effective *apertures* of the several lenses after the first, and the *visual angle;* and, finally, the last focus of this pencil is the point at which the eye must be placed so as to receive the entire extent of the field.

(318.) It is required to determine the foci of a pencil of rays proceeding from any one point of the object, or the places of the several images.

Let f, f', f'', &c. $f^{(n)}$, denote the focal lengths of the several lenses composing the system, and e, e', e'', &c. $e^{(n-1)}$, the successive intervals between them; also, let a and b denote the conjugate focal distances of the first lens, a' and b' those of the second, &c., and $a^{(n)}$ and $b^{(n)}$ those of the last, or $(n+1)$th lens. Then from (168.) we learn that these quantities are connected by the equations

$$\frac{1}{b} = \frac{1}{a} + \frac{1}{f}, \quad \frac{1}{b'} = \frac{1}{a'} + \frac{1}{f'}, \text{ \&c. } \frac{1}{b^{(n)}} = \frac{1}{a^{(n)}} + \frac{1}{f^{(n)}};$$

$$a' = b + e, \quad a'' = b' + e', \text{ \&c. } \quad a^{(n)} = b^{(n-1)} + e^{(n-1)}.$$

By elimination among these equations we shall obtain successively b, b', &c. $b^{(n)}$; and therefore the places of the several images, real or virtual.

In order that the object may be seen *distinctly* through the combination, the emergent rays must fall upon the eye with such a degree of divergence as is best suited to distinct vision. Hence, as the eyes of most persons are adapted to bring parallel rays to a focus on the retina without effort, it is usual to adjust all combinations of lenses, intended for optical instruments, in such a manner that the rays of each pencil may emerge parallel, or, in other words, that the last image may be infinitely distant. Hence $\frac{1}{b^{(n)}} = 0$, and the last equation of the first series is reduced to

$$a^{(n)} = -f^{(n)}.$$

And if we eliminate b, a', b', &c. $a^{(n)}$, from the preceding equations, the resulting equation will express the condition which must be fulfilled by the distance of the object, the focal lengths and intervals of the lenses, so that the combination may be adapted to distinct vision.

When the combination is intended for the examination of *very distant* objects, $\frac{1}{a} = 0$; and the first equation of the first series is reduced to

$$b = f.$$

(319.) In order to determine the *visual angle* under which an object is seen through the system, let θ denote the angle which the axis of the extreme pencil makes with the common axis of the system at its first intersection, namely, at the centre of the object-glass; and θ', θ'', &c. $\theta^{(n)}$, the angles which it makes with the same after refraction by the second, third, &c. $(n+1)$th lenses, respectively. Also, let c' and d' denote the distances (measured from the second lens) of the intersections of this axis with the common axis of the system, before and after refraction by the second lens; c'' and d'', c''' and d''', &c. $c^{(n)}$ and $d^{(n)}$, the analogous quantities for the third, fourth, &c. $(n+1)$th lenses respectively; then we have

$$\frac{\tan.\theta'}{\tan.\theta}=\frac{c'}{d'},\quad \frac{\tan.\theta''}{\tan.\theta'}=\frac{c''}{d''},\ \&c.\quad \frac{\tan.\theta^{(n)}}{\tan.\theta^{(n-1)}}=\frac{c^{(n)}}{d^{(n)}}.$$

And multiplying these equations together, and denoting the ratio $\frac{\tan.\theta^{(n)}}{\tan.\theta}$ by ϱ, we find

$$\varrho=\frac{c'c''\,..\,c^{(n)}}{d'd''\,..\,d^{(n)}}.$$

It is evident from what has been said (317.) that c' and d', c'' and d'', &c. $c^{(n)}$ and $d^{(n)}$, are conjugate focal distances to the several lenses, after the first or object-lens; and therefore that the relations amongst them will be given by the equations

$$\frac{1}{d'}=\frac{1}{c'}+\frac{1}{f'},\quad \frac{1}{d''}=\frac{1}{c''}+\frac{1}{f''},\ \&c.\quad \frac{1}{d^{(n)}}=\frac{1}{c^{(n)}}+\frac{1}{f^{(n)}};$$

$$c'=e,\quad c''=d'+e',\ \&c.\quad c^{(n)}=d^{(n-1)}+e^{(n-1)}.$$

And if, by means of these equations, we eliminate $\frac{c'}{d'}$, $\frac{c''}{d''}$, &c. from the value of ϱ, we shall have the ratio of the tangents of the visual angles expressed in terms of the focal lengths of the lenses and of the intervals between them.

The value of ϱ, thus obtained, is independent of any relation whatever amongst the focal lengths and intervals of the lenses. When, therefore, the combination is intended for optical purposes, it is necessary to introduce in the value of ϱ the relation amongst these quantities involved in the condition of distinct vision (318.).

The object seen through the combination will appear erect or inverted, according as θ and $\theta^{(n)}$, the angles which the axis of the extreme pencil makes with the axis of the system before and after refraction respectively, are of the same or of opposite signs; these angles being measured from the portion of the axis extending *towards* the incident light. Accordingly the *position* of the object, as seen through the instrument, will be determined by the sign of ϱ, being *erect* when ϱ is positive, *inverted* when negative.

(320.) In instruments intended for the vision of *very distant* objects, the *magnifying power* is estimated by the ratio which the visual angle, under which the object is seen by the aid of the instrument, bears to that which it subtends to the naked eye at its *actual* distance; that is, by $\frac{\theta^{(n)}}{\theta}$; since θ, the angle which the object subtends at the centre of the object-glass, is *q. p.* equal to that which it subtends to the naked eye. Wherefore, if we substitute the ratio of the tangents of these angles for that of the angles themselves (these ratios being *q. p.* equal when the angles are small), and denote the magnifying power by M, we have

$$\text{M} = \varrho.$$

In estimating the magnifying power of instruments intended for the examination of *near* objects, it is usual to compare the visual angle, under which the object is seen by the aid of the instrument, with that under which it appears to the naked eye at the *least distance of distinct vision*. Now the tangent of this latter angle is equal to

$$\frac{a}{\lambda}\tan.\theta,$$

a being the distance of the object from the object-glass, and the least distance of distinct vision; and the ratio of the tangents of the angles compared is $\frac{\lambda}{a} \cdot \frac{\tan.\theta^{(n)}}{\tan.\theta} = \frac{\lambda}{a}\varrho$. And, substituting this ratio of the tangents of the angles for that of the angles themselves, we have

$$\text{M} = \frac{\lambda}{a}\varrho.$$

(321.) In order to determine the magnifying power of a given telescope in practice, by means of the preceding equations, it becomes necessary to measure with accuracy the intervals between the several lenses and their focal lengths; and, this being attended with much difficulty, it is usual in practice to have recourse to some independent method of determining it by observation. The method usually employed for this purpose is as follows: a circular aperture, about an inch in diameter, is cut in a blackened card, and in another card a rectangular aperture is made whose breadth is exactly equal to the diameter of the former. The circular aperture then, being placed between the eye and the sky at the distance of one or two hundred yards, is viewed with one eye through the telescope; while with the other the rectangular aperture is viewed directly: this latter is then moved until them agnified image of the circle appears exactly to fill the breadth of the rectangle; and, the distances of the two cards being then measured, the quotient of the former by the latter will evidently be the magnifying power of the instrument.

A very simple and elegant method of obtaining the magnifying power of a telescope in practice is suggested by the value of ϱ given above. For it is evident, on comparing that value with the result of (196.), that ϱ is equal to the ratio which the linear magnitude of the object-glass bears to that of its last image, which is formed beyond the eye-glass at the intersection of the axis of the extreme pencil with the common axis of the lenses. Observing, therefore, the diameter of this image, and dividing by it the diameter of the object-glass already known, we shall have at once the magnifying power.

The diameter of this image may be measured by means of a thin plate of horn or mother-pearl, on which a fine scale of equal parts is drawn. This scale being brought to coincide with the image will give at once its diameter. When greater accuracy is sought, the diameter of the image may be measured by means of a double image micrometer. Such is the instrument invented by Ramsden, and which, from its application, has been called a *dynameter*. It is obvious that it cannot be applied to Galileo's telescope, or to any instrument in which the axes of the extreme pencils *diverge* at emergence.

(322.) We now proceed to compare the *quantity of light* and the *brightness* of the image of an object, seen through any combination of lenses, with those of the image of the same object seen directly.

The quantity of light incident from any distant object upon the object-glass is to that which falls from the same object upon the pupil of the eye in the ratio of the areas of the surfaces, that is, as A^2 to a^2; A and a denoting the diameters of the object-glass and of the pupil respectively. Hence the quantity of light incident upon the object-glass is equal to $\left(\frac{A}{a}\right)^2$, that which falls upon the pupil being unity; and, if m denote the ratio which the transmitted bears to the incident light, after passing through the combination, the quantity of light in the image of the object, seen through the system, will be

$$m\left(\frac{A}{a}\right)^2;$$

the quantity of light in the image on the retina of the naked eye being unity.

On the magnitude of this ratio depends what Sir William Herschel has called the telescopic *power of penetrating space.* In different telescopes, therefore, this power varies *cæt. par.* as the areas of the object-glasses.

When the combination is employed for the vision of *near* objects, the quantity of light received through it is to be compared with that received by the naked eye at the least distance of distinct vision. We must substitute, therefore, for the areas of the object-glass and pupil in the preceding investigation, these areas divided by the squares of their distances from the object, or $\left(\frac{A}{a}\right)^2$ and $\left(\frac{\alpha}{\lambda}\right)^2$, a and λ denoting as before (320.). The ratio, therefore, which the quantity of light in the image of a near object, seen through any combination of lenses, bears to that of the image on the retina of the naked eye is

$$m\left(\frac{\text{A}\,\lambda}{\alpha\,a}\right)^2.$$

The ratio above given can never exceed a certain limit dependent upon the magnifying power of the instrument. This limit is attained when

$$\frac{\text{A}}{\alpha} = \varrho;$$

that is, when $\frac{\text{A}}{\varrho} = \alpha$, or the last image of the object-glass, which is the smallest space into which the emergent rays are collected, is equal to the aperture of the pupil. For, when $\frac{\text{A}}{\alpha} > \varrho$, the smallest space into which the emergent rays are collected is greater than the aperture of the pupil; and all the rays which fall without the pupil being lost, in order to obtain the effective quantity of light, we must diminish the value above given in the ratio of the area of this smallest space to that of the aperture of the pupil, that is, in the ratio $\left(\frac{\text{A}}{\alpha}\right)^2$ to ϱ^2. Accordingly, the maximum quantity of light in the image of a distant object $= m\varrho^2$; and that in the image of a near object $= m\left(\varrho\frac{\lambda}{a}\right)^2$, that of the image on the retina of the naked eye being unity. Or, since $\text{M} = \varrho$ in the former case, and $\text{M} = \varrho\frac{\lambda}{a}$ in the latter, the maximum quantity of light in the image of an object seen through any instrument is, generally,

$$m\,\text{M}^2.$$

(323.) The *apparent brightness* is measured by the quantity of light in the image on the retina divided by the space over which it is diffused. Now it is evident that the area of the image on the retina, when the object is seen through the instrument, is equal to M^2, the area of the image on the retina of the naked eye being unity. Hence, if we divide the expression of the quantity of the light, in the case of a distant object, by ρ^2, and in the case of a near object by $\left(\rho\frac{\lambda}{a}\right)^2$, the quotient in both cases is

$$m\left(\frac{\text{A}}{\alpha\rho}\right)^2;$$

which accordingly expresses the brightness of an object seen through any combination of lenses, the brightness of the object seen with the naked eye being unity.

It is evident, from what has been said above, that this value can never exceed a certain limit, to which it attains when $\text{A} = \alpha\rho$; and therefore that the *greatest brightness* of an object, seen through any combination of lenses, is equal to

$$m,$$

that of the object seen by the naked eye being unity. Hence, m being less than unity, the brightness of an object seen through any combination of lenses is always less than that which it exhibits to the naked eye; the former, when greatest, being to the latter in the ratio of the transmitted to the incident light. In most cases, however, the brightness does not reach this limiting value; since it seldom happens that A is so great as $\alpha\rho$, except in instruments of small magnifying power.

(324.) We now proceed to determine the *apertures* of the several lenses, corresponding to a given magnitude of the *field of view.*

A', A'', &c. $\text{A}^{(n)}$ denoting the apertures of the several lenses, after the first or object lens, it is evident that they are connected by the relations

$$\frac{\text{A}''}{\text{A}'} = \frac{c''}{d'}, \quad \frac{\text{A}'''}{\text{A}''} = \frac{c'''}{d''}, \text{ \&c.} \quad \frac{\text{A}^{(n)}}{\text{A}^{(n-1)}} = \frac{c^{(n)}}{d^{(n-1)}}.$$

And multiplying we have

$$\frac{\text{A}''}{\text{A}'} = \frac{c''}{d'}, \quad \frac{\text{A}'''}{\text{A}'} = \frac{c''c'''}{d'd''}, \text{ \&c.} \quad \frac{\text{A}^{(n)}}{\text{A}'} = \frac{c''c''' \ldots c^{(n)}}{d'd'' \ldots d^{(n-1)}}.$$

Now, if Θ denote the original inclination of the axes by which these several apertures are limited, or the given field of view, $\text{A}' = e\Theta$; and we have

$$\mathrm{A}' = \alpha'\Theta, \quad \mathrm{A}'' = \alpha''\Theta, \quad \mathrm{A}''' = \alpha'''\Theta, \text{ \&c. } \quad \mathrm{A}^{(n)} = \alpha^{(n)}\Theta;$$

in which the coefficients α', α'', α''', &c. are determined by the equations

$$\alpha' = e, \quad \alpha'' = e\frac{c''}{d'}, \quad \alpha''' = e\frac{c''c'''}{d'd''}, \text{ \&c. } \quad \alpha^{(n)} = e\frac{c''c''' \ldots c^{(n)}}{d'd'' \ldots d^{(n-1)}}.$$

By these equations the apertures of the several lenses corresponding to a given magnitude of the field of view are determined. These may be called the *effective apertures.*

From the preceding equations we obtain a remarkable relation connecting the magnifying power, the field of view, and the apertures and powers of the several lenses. For, if we multiply them by the equations,

$$\frac{1}{f'} = \frac{1}{d'} - \frac{1}{c'}, \quad \frac{1}{f''} = \frac{1}{d''} - \frac{1}{c''}, \quad \frac{1}{f'''} = \frac{1}{d'''} - \frac{1}{c'''}, \text{ \&c.}$$

each by each, and substitute for e its value c', they become

$$\frac{\alpha'}{f'} = \frac{c'}{d'} - 1, \quad \frac{\alpha''}{f''} = \frac{c'}{d'}\left(\frac{c''}{d''} - 1\right), \quad \frac{\alpha'''}{f'''} = \frac{c'c''}{d'd'}\left(\frac{c''}{d'''} - 1\right),$$
$$\text{\&c.}$$

And adding them together, substituting for $\frac{c'c'' \ldots c^{(n)}}{d'd'' \ldots d^{(n)}}$ its value ϱ, and multiplying the result by Θ, we obtain

$$\frac{\mathrm{A}'}{f'} + \frac{\mathrm{A}''}{f''} + \frac{\mathrm{A}'''}{f'''} + \text{\&c.} + \frac{\mathrm{A}^{(n)}}{f^{(n)}} = (\varrho - 1)\Theta.$$

From this it appears in what manner the field of view is increased by adding to the number of lenses. It appears also that the field is diminished by increasing the magnifying power; since, *cæt. par.*, Θ varies inversely as $\varrho - 1$.

(325.) If the values of the coefficients of the apertures, α', α'', α''', &c., obtained in the preceding article, be developed by eliminating $\frac{c''}{d'}$, $\frac{c'''}{d''}$, &c. by means of the equations (319.) we obtain

$$\alpha' = e,$$

$$\alpha'' = e + e' + \frac{ee'}{f'},$$

$$\alpha''' = e + e' + e'' + \frac{e(e' + e'')}{f'} + \frac{e''(e + e')}{f''} + \frac{ee'e''}{f'f''},$$

&c. &c.;

by which the apertures of all the lenses, after the first, are determined in terms of their focal lengths, and of the intervals between them.

If any of the apertures be diminished in any ratio, it is evident that the field of view will be proportionally diminished; since the coefficients α', α'', α''', &c., depending solely on the focal lengths and positions of the lenses, will be constant when these are given. Hence, when the apertures of the lenses are given independently, the field of view, corresponding to each, will still be given by the preceding equations; and it is evident that the *actual field*, visible through the entire system, will be the least of the values of Θ thus obtained. Accordingly there will be either a loss of field, or a useless extent of aperture, when the *actual* apertures of the several lenses do not agree with the *effective;* that is, when the actual apertures do not fulfil the conditions involved in the preceding equations.

The field of view of a telescope is easily ascertained practically. We have only to direct the instrument to a star at or near the equator, and observe the time of its passage across the field. The number of seconds in this time, multiplied by 15, will give the number of seconds in the field of view; since a star in the equator has an angular motion of 15″ in a second of time.

(326.) In order that the eye may receive the entire extent of the field, it must be placed at the point in which the axes of the extreme pencils intersect the common axis of the lenses after emergence; that is, at the point whose distance from the last lens $= d^{(n)}$, the value of which is to be obtained by elimination among the equations (319.). When the aperture of the last lens, however, the magnifying power and the field of view, are already known, the value of $d^{(n)}$ is immediately obtained;

for, $\Theta^{(n)}$ being the angle made by the extreme axes at emergence, we have

$$A^{(n)} \doteq d^{(n)} \Theta^{(n)} = d^{(n)} \varrho \Theta, \quad \therefore \; d^{(n)} = \frac{A^{(n)}}{\varrho \Theta}.$$

The eye, it is evident, must always lie at the *negative* side of the eye-glass. Wherefore, when the value of $d^{(n)}$ is positive, the eye cannot be placed so as to admit the whole extent of the field; and, to give it its most advantageous position, we must place it as close as possible to the eye-glass. In this case the effective aperture of the eye-glass is reduced to that of the pupil of the eye; and we must substitute the latter for the former in the equations involving the field of view.

II.

Of the Astronomical Telescope.

(327.) If a convex lens be placed so as to receive the rays proceeding from a distant object, an image of that object will be formed at its principal focus, which may be viewed directly by an eye placed beyond it at a distance not less than the least distance of distinct vision.

Let m denote the linear magnitude of this image, and λ the least distance of distinct vision: the angle which the image will subtend to the eye placed at that distance will be $\frac{m}{\lambda}$. But the angle which the object subtends at the centre of the lens (or at the eye, *q.p.*) is equal to that which its image subtends at the same place, with the opposite sign, or to $\frac{-m}{f}$, f denoting the focal length of the lens. And the former angle is to the latter in the ratio of f to λ; or, in other words, the magnifying power of the instrument is expressed by the fraction

$$\frac{-f}{\lambda}.$$

From which it is evident that there will be an increase of the visual angle, whenever the focal length of the lens is greater than the least distance of distinct vision. This may be considered as a telescope in its simplest form.

But if, instead of viewing this image directly, it be viewed by means of a second convex lens, placed at a distance from that image equal to its own focal length, f', it will be seen under an angle equal to $\frac{m}{f'}$ (301.), whatever be the position of the eye; and the ratio of this angle to that which the object subtends to the unassisted eye, or the magnifying power of the combination, is

$$\text{M} = \frac{-f}{f'}.$$

Accordingly, when the focal length of the second lens is less than the least distance of distinct vision, the magnifying power is increased by the addition; and, since the rays emerge parallel, they will be brought to a focus on the retina without effort*.

(328.) Such is the *common astronomical telescope*, the construction of which was first explained by the famous Kepler.

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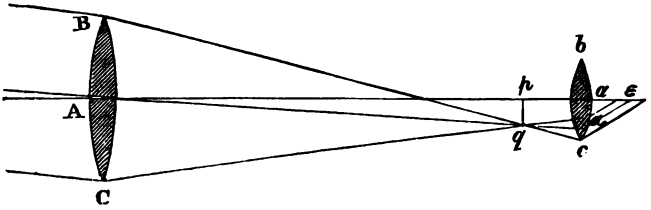
* If the second lens be brought somewhat nearer to the image formed by the first, so that the rays may emerge from it diverging from a second image, whose distance from the lens is equal to λ, the least distance of distinct vision, that image will be viewed by the eye, placed close to the lens, under an angle equal to $m\left(\frac{1}{\lambda}+\frac{1}{f'}\right)$ (300.); and the magnifying power of the combination becomes

$$\text{M} = -f\left(\frac{1}{f'}+\frac{1}{\lambda}\right).$$

In this case, however, the rays are incident upon the eye diverging from the least distance admitting distinct vision.

It consists, accordingly, of two convex lenses, as BC and *bc* in the adjoining figure, fixed at the extremities of a tube, at a



distance equal to the sum of their focal lengths. The lens BC, which is turned towards the object, is called the *object-glass*, and has a large aperture and considerable focal length; that turned towards the eye, *bc*, is called the *eye-glass*, and is of small aperture and short focal distance. An inverted image of the object, *pq*, is formed at the common focus of the two lenses. B*q*, A*q*, and C*q*, are three rays of a pencil which diverge from any one point of the object, and by the refraction of the object-glass are made to converge to *q*, the corresponding point of the image. These rays, crossing at this point, are incident upon the eye-glass diverging, and by its refraction are made to emerge parallel. The ray Aαε, which passes through the centre of the object-glass, is the axis of this pencil; and the ratio of the angles, αA*a*, αε*a*, which it makes with the axis of the lenses before and after refraction by the eye-glass respectively, determines the magnifying power of the telescope.

Since Aαε, the axis of the pencil proceeding from the upper part of the object, crosses the axis of vision again at ε, and therefore proceeds to the upper part of the eye, the image on the retina will be *erect* with respect to the object; and, consequently, the object will appear in the opposite position to that which it has to the naked eye (288.), or *inverted*. This inversion of the object, as seen through the astronomical telescope, is also indicated by the negative sign of the magnifying power (319.).

From what has been said, it is evident in what manner this instrument assists the eye in the vision of distant objects. The object-glass, being much broader than the pupil, receives a much larger portion of the light flowing from the object than the unassisted eye can do, and thus aids the vision by increasing the quantity of light. These rays, diverging from the several points of the object, are made to converge to the principal focus of the object-glass, and form an image there; and the eye-glass enables us to view this image under the same visual angle as if we approached it to the distance of the focal length of that glass, and with all the circumstances necessary for distinct vision.

(329.) If this instrument be employed for the vision of nearer objects, the angle which the object subtends at the centre of the object-glass $= -\frac{m}{b} = m\left(\frac{1}{f} - \frac{1}{a}\right)$, a and b denoting the distances of the object and image, respectively, from the object-glass. Therefore the magnifying power in this case will be

$$\text{M} = -\frac{b}{f'} = \frac{f}{f'}\left(\frac{a}{a-f}\right).$$

So that the magnifying power of the telescope is increased by accommodating it to nearer objects in the ratio of a to $a-f$. The adaptation is performed by increasing the distance of the eye-glass from the object-glass, that distance being equal to $b+f'$.

For very near objects the focal length of the object-glass must be greatly diminished; for, in order that there should be a real image formed by that lens, it is necessary that the distance of the object from the object-glass should be greater than its focal length. We have thus the construction of the *compound microscope*, which differs therefore from the astronomical telescope merely in the use of an object-glass of high power.

(330.) It is evident that the magnifying power of the common astronomical telescope is to be increased, either by increasing the focal length of the object-glass, or by diminishing that of the eye-glass. The latter of these methods, however—namely, that of diminishing the focal length or increasing the power of the eye-glass—was soon found to be incompatible with the perfection of the instrument, when adopted to any considerable extent. For the image in the focus of the object-glass being confused by the aberration of that lens, when it is viewed through an eye-glass of high power, the confusion will be increased so much as to render the vision very indistinct.

To consider this more fully, it is to be observed, that the *confusion* or indistinctness of vision will depend upon the magnitude of the circle of aberration on the retina; the diameter of which is proportional to the angle which the diameter of the least circle of aberration in the last image subtends to the eye, or to the eye-glass, *q.p.* Now, to determine this angle, let a and b, a' and b', denote the conjugate focal distances to any two lenses, A the semi-aperture of the first, and ρ the radius of the least circle of spherical aberration in the image formed by the second; then (189.) we have

$$\rho = \tfrac{1}{4}\frac{a'}{b}\mathrm{K}b'\mathrm{A}^3, \quad \text{and} \therefore \frac{2\rho}{b'} = \tfrac{1}{2}\frac{a'}{b}\mathrm{K}\mathrm{A}^3,$$

$\dfrac{2\rho}{b'}$ being the angle subtended by the diameter of the least circle of aberration at the centre of the second lens. But

$$-\mathrm{K} = \left(\frac{b}{a'}\right)^2 \varkappa + \left(\frac{a'}{b}\right)^2 \varkappa',$$

in which $\varkappa$ and $\varkappa'$ are the coefficients of the square of the aperture in the values of $d\alpha'$ for each of the lenses (181.). Wherefore, substituting, the angle of aberration is equal to

$$-\tfrac{1}{2}\frac{a'}{b}\left\{\left(\frac{b}{a'}\right)^2 \varkappa + \left(\frac{a'}{b}\right)^2 \varkappa'\right\}\mathrm{A}^3.$$

But when the two lenses form a telescope, we have

$$b = -f, \quad a' = f'.$$

Again: it is evident from the reasoning employed (312.), that

$$\varkappa = \frac{\mathrm{L}}{f^3}, \quad \varkappa' = \frac{\mathrm{L}'}{f'^3};$$

in which the quantities L and L′ depend upon the form and material of the lenses. Wherefore, making these substitutions in the preceding expression, it is reduced to

$$\left(\frac{\text{L}}{f'} + \frac{\text{L}'}{f}\right)\frac{\text{A}^3}{2f^2}:$$

or, if we neglect the second term within the brackets in comparison with the first on account of the magnitude of f compared with that of f', the angle of aberration becomes

$$\tfrac{1}{2}\text{L}\,\frac{\text{A}^3}{f^2 f'},$$

very nearly. It varies therefore, *cæt. par.*, as $\frac{\text{A}^3}{f^2 f'}$; and the *confusion* or indistinctness of vision, which is measured by the area of the circle of aberration on the retina, will be as the square of this quantity.

(331.) But the chief source of indistinctness in the first refracting telescopes was the *chromatic aberration* of the lenses, that arising from their spherical figure being inconsiderable in comparison. Now the diameter of the least circle of chromatic aberration in the image formed by the second lens is equal to $\text{A}'\frac{\Delta b'}{b'}$ (258.); A′ being the semi-aperture of that lens, and $\Delta b'$ the chromatic variation of the distance of the image from it. Wherefore the angle subtended by the diameter of this circle at the centre of the second lens is equal to

$$\text{A}'\frac{\Delta b'}{b'^2} = -\,\text{A}'\Delta\left(\frac{1}{b'}\right).$$

But from (262.) we have, in the case of two lenses,

$$\Delta\left(\frac{1}{b'}\right) = \left(\frac{b}{a'}\right)^2\frac{\pi}{f} + \frac{\pi'}{f'},$$

π and π' denoting the dispersive powers of the substances of which the lenses are composed. Also $A' = \left(\frac{a'}{b}\right)A$; wherefore the angle of dispersion is equal to

$$-A\left\{\left(\frac{b}{a'}\right)\frac{\pi}{f}+\left(\frac{a'}{b}\right)\frac{\pi'}{f'}\right\}.$$

When $b = -f$, and $a' = f'$, as in the case of the common astronomical telescope, this expression becomes

$$A\left(\frac{\pi}{f'}+\frac{\pi'}{f}\right);$$

or, if we neglect the second term in comparison with the first, on account of the magnitude of f compared with that of f', the angle of dispersion will be equal to

$$\pi . \frac{A}{f'},$$

very nearly. Hence the diameter of the circle of aberration on the retina, which is proportional to this angle, varies as $\frac{A}{f'}$, *cæt. par.*; and the confusion of vision, which is measured by the area of that circle, will be as the square of this quantity.

To attain a high magnifying power, therefore, and at the same time to preserve sufficient distinctness in the performance of the instrument, it became necessary to increase considerably the focal length of the object-glass, and therefore also the length of the instrument. The difficulty of managing instruments of considerable length, and their liability to bend, suggested the idea of separating the object-glass from the eye-glass, and attaching it to the top of a high pole with an apparatus annexed for the purpose of varying its position. Such instruments were called *aerial* telescopes, and were those employed by Huyghens in his astronomical observations. His great telescope was 123 feet in length; that employed by the astronomer Cassini extended to the prodigious length of 150 feet.

(332.) But in whatever way we increase the magnifying power of this instrument, unless the aperture of the object-glass be proportionally increased, we diminish at the same time the *apparent brightness*, which is equal to

$$=\left(\frac{\text{A}}{\text{M}\alpha}\right)^2,$$

the brightness of the object seen by the naked eye being unity. And, by increasing the aperture of the object-glass, we increase the confusion, which varies, *cæt. par.*, as the area of that lens. Hence the improvement of the instrument, in one respect, injures its performance in another; and it becomes, therefore, of importance to determine to what extent these incompatible qualities of brightness and distinctness are to be carried so as to give, on the whole, the best performance to the instrument. Now, this can only be ascertained by experience and trial; and all that theory can do is to determine the dimensions of the several parts of a telescope, which shall perform with the same degree of brightness and distinctness as the instrument in which these qualities are found, by trial, to be combined in the best manner.

The *brightness*, we have seen, varies in the duplicate ratio of the aperture of the object-glass divided by the magnifying power; and the *indistinctness* arising from chromatic aberration varies in the duplicate ratio of the same aperture divided by the focal length of the eye-glass. Wherefore if, for the present, F and f denote the focal lengths of the object-glass and eye-glass, and A the aperture of the former, in the telescope whose dimensions are sought; F$'$, f', and A$'$, the corresponding quantities in the *standard telescope* with which it is compared; in order that the brightness and distinctness should be the same in the two instruments, we must have

$$\text{A}\frac{f}{\text{F}}=\text{A}'\frac{f'}{\text{F}'},\qquad \frac{\text{A}}{f}=\frac{\text{A}'}{f'};$$

and, if we multiply these equations together, we have

$$\frac{\text{A}^2}{\text{F}}=\frac{\text{A}'^2}{\text{F}'},\qquad \text{or } \text{A}=\text{A}'\sqrt{\frac{\text{F}}{\text{F}'}};$$

and, combining this result with the two preceding,

$$f=f'\frac{\text{A}}{\text{A}'}=f'\sqrt{\frac{\text{F}}{\text{F}'}},\qquad \text{M}=\text{M}'\frac{\text{A}}{\text{A}'}=\text{M}'\sqrt{\frac{\text{F}}{\text{F}'}}.$$

From which it appears that, in order that the brightness and distinctness should be the same in two telescopes, the apertures of the object-glasses, the focal lengths of the eye-glasses, and also the magnifying powers, must be in the sub-duplicate ratio of the focal lengths of the object-glasses, or of the lengths of the telescopes, *q. p.*

In the standard telescope of Huyghens, the focal length of the object-glass was 30 feet, its aperture three inches, and the focal length of the eye-glass three inches and three-tenths. Hence making $F' = 30$, $A' = 3$, and $f' = 3.3$, in the preceding results, we find

$$A = \sqrt{.3F}, \qquad f = 1.1A.$$

Wherefore, if the focal length of the object-glass, or the length (in feet) of the telescope whose other dimensions are required, be multiplied by the decimal .3, the square root of the product will be the aperture of the object-glass, in inches. And, if this aperture be increased by its tenth part, the result will be the focal length of the eye-glass of the telescope required. Such is, in other words, the Huyghenian rule.

(333.) The dimensions here given are those of a telescope intended for ordinary astronomical observations: but as the use of the instrument is varied, so also the proportions will vary from those here assigned. Thus in day observations, in which the light of the stars is obscured by that of the atmosphere, a greater degree of brightness is required; and this will be attained, at the expense of magnifying power, by increasing the focal length of the eye-glass. By this means, it is evident, the confusion arising from chromatic aberration is also diminished; so that it might be supposed that we could increase the aperture of the object-glass in the same proportion, and thus increase the brightness in a higher proportion, the indistinctness remaining the same as before. This, however, is found not to be the case; for, though the coloured fringes arising from chromatic aberration preserve the same dimensions as before, by a proportionate increase of the aperture of the object-glass, and of the focal length of the eye-glass, yet they become much more sensible on account of the increase of light. The aperture of the object-glass, therefore, is not to be increased beyond the dimensions assigned above, unless in the examination of objects of very faint light, such as the smaller fixed stars and the secondary planets.

On the other hand, in fitting up a telescope for the examination of a very bright object, such as the moon, or Venus, we may increase the magnifying power, at the expense of the brightness, by diminishing the aperture of the object-glass and the focal length of the eye-glass in the same proportion; for by this means the confusion remains the same, while the magnifying power is increased. This, however, cannot be done beyond certain limits; for it is found by experience that the vision is confused if the cylinder of rays, which emerge from any point of the image through the eye-glass, be less in breadth than the $\frac{1}{72}$ of an inch. Now the breadth of these cylinders is to the aperture of the object-glass as the focal length of the eye-glass is to that of the object-glass: it is therefore equal to $\mathrm{A}\frac{f}{\mathrm{F}}$, and varies in the sub-duplicate ratio of the brightness. This breadth, accordingly, remains unaltered by Huyghens' rule.

(334.) We have now seen that in order to attain a high magnifying power, consistently with a moderate degree of distinctness in the performance of the instrument, it became necessary to increase considerably the focal length of the object-glass, and therefore that of the instrument itself. The great inconvenience, however, attending the management of instruments, such as have been just described, was obvious; but it was equally evident that, as long as the telescope consisted but of a single object-glass and a single eye-glass, there was no remedy. In this manner the perfection of the telescope seemed to be limited by the effect of chromatic aberration.

It occurred to Newton, however, that the aberration of the object-lens might possibly be counteracted, or at least diminished, by the opposite aberration of a concave lens of a different refracting material, which, with a different total refraction, should have the same dispersion. To examine this point he instituted the fallacious experiment of the two prisms, already mentioned (248.), which led him to decide this question in the negative, and accordingly made him despair of any further improvement in refracting telescopes. The supposed achromatism of the eye led Euler afterwards to conceive the possibility of destroying the dispersion by the union of lenses of different substances; and having assumed a particular law of dispersion, he calculated the curvatures of the surfaces of a compound lens, consisting of water enclosed within two lenses of glass, so that the dispersion of the whole might be nothing.

The truth of these results of Euler, together with that of the law of dispersion on which they were founded, was warmly disputed by Dollond, who opposed to them the authority and the experiments of Newton: and it was not until some years afterwards, on the occasion of a paper which he received from M. Kliengstierna, professor of mathematics at Upsal, disputing the truth of the Newtonian law, that he so far doubted the accuracy of that great man's experiment as to repeat it himself. The result of this experiment, the reverse of that of Newton (248.), immediately convinced him of the practicability of correcting the dispersion in a compound object-glass composed of lenses of different materials; and he hastened to apply the principle, as Euler had before endeavoured to do, in the construction of an achromatic lens composed of water enclosed between two lenses of glass. The small difference which existed between the dispersive powers of water and the glass which he employed, however, required that the component lenses should have considerable powers, and therefore that the curvatures of their surfaces should be so great as to entail a very large spherical aberration. Despairing, therefore, of forming a perfect compound lens of this construction, he was obliged to turn his attention to other refracting substances, and it happily occurred to him that the difference in the performance of the different kinds of glass, with which he was already familiar, arose from a difference in their dispersive powers, such as it was now the object to obtain. This conjecture was verified on trial, and he finally succeeded in constructing an achromatic compound lens, consisting of a convex lens of *crown glass* and a concave of *flint glass*, similar in all respects to those at present in use.

We have already shown (185.-6.) (264.) in what manner a compound lens may be constructed which shall be free, or nearly so, from all errors, whether arising from the sphericity of the surfaces or from the unequal refrangibility of light; and such a lens being employed as the object-glass of a telescope, the perfection of the image produced by it enables us to employ an eye-glass of high power, and thus to increase the power of the instrument without an undue increase of length. In this manner the perfection of the instrument is limited only by the errors arising from the eye-glass, which are comparatively small. In all good instruments, however, instead of a single eye-glass, a compound eye-piece composed of several lenses is employed, in which the errors of dispersion and form are in a great degree removed*.

(335.) The *field of view*, or the angular extent of any object visible through the telescope, is defined by the axes of the extreme pencils which are transmitted by the eye-glass; being equal to the angle contained by these axes, or to the angle which the aperture of the eye-glass subtends at the centre of the object-glass. Wherefore, if A' denote the semi-aperture of the eye-glass, and Θ half the field of view,

$$\Theta = \frac{A'}{f+f'}.$$

In order to take in the whole extent of this field, the eye must be placed at the point in which the axes of the extreme pencils, diverging from the centre of the object-glass, intersect the common axis of the lenses after refraction by the eye-glass. The place of the eye, therefore, is the focus conjugate to the centre of the object-glass, or the point in which an image of the object-glass will be formed by the refraction of the eye-glass (317.). But if c' and d' denote the distances of the object-glass and its image, respectively, from the eye-glass, we have

$$\frac{1}{d'} = \frac{1}{c'} - \frac{1}{f'};$$

whence, substituting for c' its value $f+f'$, we obtain

$$d' = -\frac{f'}{f}(f+f').$$

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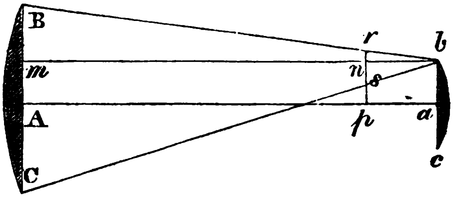
* See the 4th section of this chapter.

That is, the distance of the eye from the eye-glass must be a fourth proportional to the focal length of the object-glass, the focal length of the eye-glass, and the sum of these focal lengths.

The place of the eye is always attended to in the construction of the telescope. The tube containing the eye-glass is prolonged to the required distance, and there furnished with an eye-stop, by which means the eye is in its proper position when applied close to the extremity of the instrument.

(*336.*) The field of view, as above determined, is the angular extent of the field, the *axes* of the pencils flowing from the extreme points of which are transmitted by the eye-glass; and thus understood, we have seen, the field is altogether independent of the magnitude of the object-glass. This, however, is not the entire extent of the visible field, and to determine the magnitude of the field from which *any rays* whatever are transmitted by the two glasses, we have only to join the corresponding extremities of the two lenses by the line B*b*; *pr*, the intercepted portion of the perpendicular erected at the common focus of the two lenses, will be the semi-diameter of the entire visible extent of the image, and the angle which it subtends at the centre of the object-glass measures the *entire visible field*. To determine this angle in terms of the apertures and focal lengths of the two lenses, let the line *mnb* be drawn through the extremity of the eye-glass parallel to the axis of the telescope; then, by similar triangles, we have



$$\frac{\mathrm{B}m}{mb}=\frac{rn}{nb}, \quad \text{or} \quad \frac{\mathrm{A}-\mathrm{A}'}{f+f'}=\frac{m'-\mathrm{A}'}{f'};$$

A and A$'$ denoting the semi-apertures of the two lenses, and m' half the linear magnitude of the image. Hence

$$m' = \frac{Af' + A'f}{f + f'}.$$

But $m' = f\Theta'$, Θ' denoting the angle *p*A*r*, or half the visible field; wherefore we have

$$\Theta' = \frac{A' + \frac{f'}{f}A}{f + f'}.$$

(337.) Towards the edges of this field the light will be very faint: for it is evident that the extreme point, *r*, of the image, as seen through the eye-glass, is illuminated but by a single ray of the pencil flowing from the extreme point of the object, and incident upon the object-glass; since any other ray of that pencil will cross the extreme ray B*b* in *r*, and therefore never meet the eye-glass. And as the extreme points of the image are illuminated by but a single ray, so the neighbouring points, as they recede from the edge, receive more and more of the pencils flowing from the corresponding points of the object; and thus the image becomes gradually brighter up to a certain point, beyond which the points of the image are illuminated by the *whole* of the pencils proceeding from the corresponding points of the object. The portion of the image included within these points determines the *bright part* of the field of view, and is evidently of uniform brightness. Its magnitude is determined by connecting the opposite extremities of the two lenses by the line *cb* (see figure in preceding page): the included portion of the image, *ps*, is evidently illuminated by the whole of each incident pencil. To determine its magnitude we have in the similar triangles *cmb*, *snb*,

$$\frac{cm}{mb} = \frac{sn}{nb}, \quad \text{or} \quad \frac{A + A'}{f + f'} = \frac{A' - m''}{f'};$$

m'' denoting the semi-diameter of the bright part of the image. Wherefore

$$m'' = \frac{A'f - Af'}{f + f'}.$$

But $m'' = f\theta''$, θ'' denoting the angular magnitude of the bright part of the field; wherefore we have

$$\theta'' = \frac{A' - \frac{f'}{f}A}{f + f'}.$$

When $\frac{A'}{A} = \frac{f'}{f}$, that is, when the apertures of the lenses are as their focal lengths, θ'' vanishes; and in this case, accordingly, the brightness of the field decreases from the centre to the circumference. When $\frac{A'}{A} < \frac{f'}{f}$, the value of θ'' becomes negative, and no part of the field will be illuminated by the whole of the pencil proceeding from the corresponding point of the object.

If the values of θ' and θ'' be added together, we find

$$\theta' + \theta'' = \frac{2A'}{f + f'} = 2\theta.$$

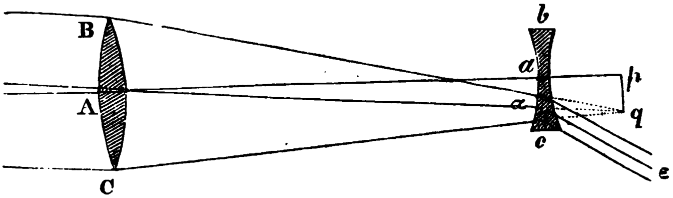
From which we learn that the field, which has been estimated by the inclination of the extreme axes, is an arithmetical mean between the *extreme field* and the *bright part* of the field.

III.

Of the Galilean Telescope.

(338.) This telescope, called after its inventor Galileo, was the first whose construction was explained on theoretical principles. It differs from the common astronomical telescope merely in the *form* and *position* of the eye-glass. This, instead of being convex, is *concave;* and is placed *between* the object-glass and its principal focus, at a distance from the latter equal to its own focal length. In this position, accordingly, the rays of each pencil converge to the principal focus of rays proceeding in an opposite direction, and consequently, emerging parallel, are in a state suited to distinct vision.

It is represented in the annexed figure, in which BC represents the object-glass, and *bc* the eye-glass. *pq* is an inverted image of the object formed in the common focus of the two lenses. B*q*, A*q*, and C*q*, are three rays of a pencil which proceed from any one point of the object, and by the refraction of the object-glass are made to converge to *q*, the corresponding point of the image; these rays, being incident upon the eye-glass converging to the principal focus of rays proceeding in the contrary direction, are by its refraction made to emerge parallel; and therefore will be brought to a focus on the retina without effort.



(339.) The image of the object, which is formed at the principal focus of the object-glass, being also in the principal focus of the eye-glass, the angle under which it is seen is equal to $\frac{-m}{f'}$, whatever be the position of the eye (301.); m being the linear magnitude of that image, and f' the focal length of the eye-glass. But the angle which the object subtends at the centre of the object-glass (which is equal to that which it subtends to the naked eye, *q. p.*) is equal to the angle subtended by its image at the same place, or equal to $\frac{-m}{f}$, f being the focal length of the object-glass. Wherefore the ratio of these angles, or the magnifying power of the telescope, is

$$\text{M} = \frac{f}{f'};$$

as in the common astronomical telescope.

Hence, if the two lenses be of the same power in the common astronomical and Galilean telescopes, the magnifying powers of the two instruments will be the same. The latter, however, will have the advantage of being shorter than the other, its length being equal to the *difference* of the focal lengths of the two lenses; whereas in the astronomical telescope it is their *sum*.

Another and more important advantage which this instrument possesses is, that objects seen through it appear *erect*, instead of being inverted, as in the common astronomical telescope. This will easily appear if we consider that the axes of the extreme pencils, being incident diverging upon the eye-glass, will diverge more after refraction by it. Consequently the pencil which flows from the uppermost part of the object proceeds to the lowermost part of the retina, and *vice versâ;* and therefore the object is seen in the same position as it would appear to the naked eye. This advantage, combined with its shortness, renders it a very convenient instrument. Accordingly we find that most telescopes of small power, such as the common *opera-glass*, &c. are made of this construction.

(340.) The *field of view* in this instrument is very limited. For the axes of the pencils which flow from the several points of the object, diverging from the centre of the object-glass, will diverge more after refraction by the concave eye-glass, and will, therefore, for the greater part, fall without the pupil of the eye, and thus be lost. In order that the eye may receive as wide an extent of field as possible, it must be placed as near as possible to the point from which these axes diverge, and therefore as close as possible to the eye-glass. In this position of the eye the effective aperture of the eye-glass is reduced to that of the *pupil;* and therefore the field of view, which is measured by the inclination of the axes of the extreme pencils which enter the eye, is equal to the angle which the aperture of the pupil subtends at the centre of the object-glass, or at a distance equal to the length of the telescope.

(341.) The preceding results are readily deduced from the general theory. For c' and d' denoting the distances of the intersections of the axes of the several pencils from the eye-glass, before and after refraction by that glass respectively, we have

$$\frac{1}{d'} = \frac{1}{c'} + \frac{1}{f'}.$$

Hence the magnifying power is

$$\text{M} = \frac{c'}{d'} = \frac{c' + f'}{f'} = \frac{f}{f'};$$

since $c' = e = f - f'$. And this value of the magnifying power being *positive*, it follows that the object will appear *erect* (319.).

Again, the distance of the eye from the eye-glass, so as to receive the entire field, must be

$$d' = \frac{c'f'}{c' + f'} = \frac{f'}{f}(f - f')^*.$$

But this value being *positive*, the eye cannot be placed in the required position; and, that it may receive as wide an extent of field as possible, it must be placed close to the eye-glass (326.). Thus the effective aperture of the eye-glass is reduced to that of the pupil of the eye; and, denoting this by α, as before, we have

$$\alpha = e\Theta = (f - f')\Theta.$$

(342.) Such is the value of the field of view, limited by the *extreme axes* which enter the eye diverging from the centre of the object-glass. The extent of the field, however, from which *any rays* whatever reach the eye, is greater than this. It is determined by connecting the opposite extremities of the object-glass and the pupil, BC and $\beta\gamma$; the line Cβr, produced to meet the perpendicular raised at p, the common focus of the object-glass and eye-glass, will determine the portion of the image, *pr*, corresponding to the extent of the object from which

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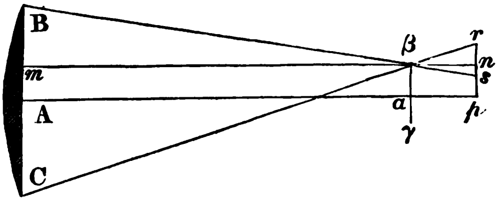
* This and the preceding result might have been at once obtained from the analogous results in the case of the common astronomical telescope (327. 335.), by simply changing the sign of f', the focal length of the eye-glass.



any rays whatever reach the eye. Now if the line $m\beta n$ be drawn through the extremity of the pupil, parallel to the axis of the telescope, in the similar triangles $\mathrm{C}m\beta$, $rn\beta$, we have

$$\frac{\mathrm{C}m}{m\beta}=\frac{rn}{n\beta}, \quad \text{or } \frac{\mathrm{A}+\alpha}{f-f'}=\frac{m'-\alpha}{f'};$$

A denoting the semi-aperture of the object-glass, α that of the pupil, and m' the linear magnitude of the image pr. Hence we have

$$m'=\frac{\mathrm{A}f'+\alpha f}{f-f'};$$

but if Θ' denote the magnitude of the field corresponding to m', $m'=f\Theta'$, and therefore

$$\Theta'=\frac{\alpha+\dfrac{f'}{f}\mathrm{A}}{f-f'}.$$

In like manner the bright part of the field is determined by connecting the adjacent extremities of the object-glass and the pupil by the line $\mathrm{B}\beta s$; the angle which the intercepted portion of the image, ps, subtends at the centre of the object-glass is the bright part of the field. Denoting it by Θ'', and the corresponding portion of the image, ps, by m'', in the similar triangles $\mathrm{B}m\beta$, $sn\beta$, we have

$$\frac{\mathrm{B}m}{m\beta}=\frac{sn}{n\beta}, \quad \text{or } \frac{\mathrm{A}-\alpha}{f-f'}=\frac{\alpha-m''}{f'};$$

from which we obtain

$$m'' = \frac{\alpha f - \text{A}f'}{f - f'}, \text{ and } \Theta'' = \frac{\alpha - \frac{f'}{f}\text{A}}{f - f'}.$$

From this result we learn that $\Theta'' = 0$, when $\text{A} = \frac{f}{f'}\alpha$. That is, when the aperture of the object-glass is equal to that of the pupil multiplied by the magnifying power, the bright part of the field is reduced to a point at the centre, and the brightness decreases gradually from the centre to the circumference. If $\text{A} > \frac{f}{f'}\alpha$, no part of the field is illuminated by the whole of the pencil proceeding from the corresponding point of the object.

If the values of Θ' and Θ'' be added together, we find

$$\Theta' + \Theta'' = \frac{2\alpha}{f - f'} = 2\Theta.$$

That is, the field determined by the extreme axes, is an arithmetical mean between the *extreme field* and the *bright part* of the field.

(343.) The angle which the diameter of the circle of aberration from sphericity subtends to the eye, in this telescope, is

$$\left(\frac{\text{L}'}{f} - \frac{\text{L}}{f'}\right)\frac{\text{A}^3}{2f^2};$$

as will appear by changing the sign of f' in the expression obtained (330.). From this it appears that the aberration of the concave eye-glass tends to correct that produced by the object-glass, the two terms of this expression being of opposite signs.

If the curvatures of the surfaces be equal in the lenses of the *common astronomical* and *Galilean* telescopes, it is evident from the equations of (172. 3.) that the values of L and L' will be the same in the two instruments; consequently the angle of aberration in the Galilean telescope is less than that in the common astronomical, for the same aperture, in the ratio of $\text{L}f - \text{L}'f'$ to $\text{L}f + \text{L}'f'$; and the confusion arising from spherical aberration will be less in the duplicate ratio.

(344.) Again, if we change the sign of f' in the expression obtained (331.), we find that the angle which the diameter of the circle of chromatic aberration subtends to the eye, in this instrument, is equal to

$$A\left(\frac{\pi'}{f} - \frac{\pi}{f'}\right);$$

π and π' being the dispersive powers of the two lenses. Hence it appears that, as in the former case, the concave eye-glass tends to correct the aberration produced by the object-glass. And comparing the common astronomical and Galilean telescopes, in which the two lenses are of the same material and of equal power, we find that the angle of aberration in the latter is less than in the former in the ratio of $\pi f - \pi' f'$ to $\pi f + \pi' f'$. And the confusion arising from chromatic aberration will be less in the duplicate of that ratio.

The chromatic aberration will be reduced to nothing when

$$\frac{f}{f'} = \frac{\pi'}{\pi};$$

that is, when the focal lengths of the object-glass and eye-glass are to one another inversely as the dispersive powers of the substances of which they are composed. When the lenses fulfil this condition, the instrument becomes perfectly *achromatic*, as far as the central pencil is concerned, without the aid of an additional lens.

This theorem, though known to D'Alembert, was long barren of any practical consequence. For the magnifying power of the telescope thus constructed, it is evident, is equal to the ratio of the dispersive powers of the substances employed; and in *crown-glass* and *flint-glass*, the only substances whose dispersive powers engaged the attention of the earlier opticians, this ratio, at the highest, does not exceed $1\frac{1}{2}$. A magnifying power so limited could be of no value. But the great disparity in the dispersive powers of some of the substances which have formed the subjects of later experiments, suggested to Dr. Brewster the possibility of employing this construction in opera-glasses and other instruments in which a small power was required. Thus, if a telescope be constructed whose object-glass is of *rock-crystal*, and eye-glass a fluid lens of *oil of cassia*, the instrument will be achromatic if the focal lengths of the lenses be in the ratio of .139 to .026, the dispersive powers of oil of cassia and rock-crystal respectively; and the magnifying power of the instrument will be about $5\frac{1}{3}$, which is higher than what is required in an opera-glass. The ratio of the dispersive powers of *oil of cassia* and *crown-glass*, and therefore the magnifying power of the telescope constructed of them on this principle, is about 4. In *flint-glass* and *rock-crystal* it is equal to 2 very nearly. It is evident that the eye-glass must be of the substance whose dispersive power is the greater.

IV.

Of Telescopes with Compound Eye-Pieces.

(345.) It has been shown that the errors of the object-glass, both of form and colour, may be removed by combining two or three lenses together of proper form and material; and, the image produced by such a compound object-glass being free from all error, the perfection of the telescope will be limited only by the errors of the eye-glass. Now these errors, it will be easily understood from what has been said of the single microscope, cannot be removed to any extent in the single eye-glass without such a diminution either in aperture or power as would render it useless. In all good telescopes, consequently, it is usual to employ, instead of the single eye-glass, a *compound eye-piece* consisting of two or more lenses disposed at suitable intervals: and as the nature and perfection of the instrument will depend upon the adjustment of these lenses, as to power, position, and form, it will be necessary to lay down a few of the general principles of such arrangements, and to notice some of the ordinary combinations.

(346.) To begin with the simplest case: let it be proposed to examine the construction and theory of a telescope composed of three * lenses disposed in any manner along the same axis.

The relation which must subsist among the focal lengths and the positions of any three lenses, in order that a distant object may be seen distinctly through them, is contained in the equations

$$b = -f, \quad \frac{1}{a'} - \frac{1}{b'} = \frac{1}{f'}, \quad a'' = f'',$$

$$a' = b + e, \quad a'' = b' + e';$$

the lenses being supposed to be, as they usually are, *convex*, and therefore f, f', and f'', *negative* in the equations (318.). Now, from the first and third of these equations combined with the fourth and fifth, we obtain $a' = e - f$, $b' = f'' - e'$; and, these values being substituted in the second, we have finally

$$\frac{1}{e - f} + \frac{1}{e' - f''} = \frac{1}{f'};$$

the equation of condition required.

If we take away the denominators from this equation, and add f'^2 to both sides, it will assume the form

$$(f + f' - e)(f' + f'' - e') = f'^2.$$

(347.) The ratio of the visual angles, or the *magnifying power*, is

$$g = \frac{c'c''}{d'd''};$$

in which the quantities c', d', c'', d'', are determined by means of the equations (319.). Performing the elimination by means of these equations, in which we must observe to take f, f', and f'' negative,

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* Here the object-lens, if compound, is considered as a single lens.

$$\frac{c'c''}{d'd''} = \frac{c'}{d'}\left(1 - \frac{c''}{f''}\right) = \frac{c'}{d'}\left(1 - \frac{e' + d'}{f''}\right) = \frac{c'}{d'}\left(1 - \frac{e'}{f''}\right) - \frac{c'}{f''}$$

$$= \left(1 - \frac{e}{f'}\right)\left(1 - \frac{e'}{f''}\right) - \frac{e}{f''};$$

and multiplying by $f'f''$, we have finally

$$\varrho f'f'' = (e - f')(e' - f'') - ef'.$$

To introduce in this result the condition of distinct vision, we must eliminate $e' - f''$ by means of the equation of condition obtained above, and thus the preceding value of ϱ becomes

$$\varrho = -\frac{f}{f''} \cdot \frac{f'}{f + f' - e}.$$

Or, eliminating $f + f' - e$ by means of the same equation,

$$\varrho = -\frac{f}{f'f''}(f' + f'' - e').$$

From the former of these values it appears that the magnifying power is increased or diminished by the interposition of the additional lens, according as e is greater or less than f; that is, according as the distance of the second lens from the first is greater or less than the focal length of the latter.

The object will be inverted in this combination, as in the common astronomical telescope, unless when $e > f + f'$; that is, unless the distance between the first and second lenses be greater than the sum of their focal lengths.

If the magnifying power be given, as well as the focal lengths of the three lenses, the intervals between them are determined; for, from the preceding values of ϱ, we have

$$e = f + f' + \frac{ff'}{f''\varrho}, \quad e' = f' + f'' + \frac{f'f''\varrho}{f}.$$

(348.) The *apertures* of the two lenses, corresponding to a given extent of *field of view*, are (324, 5.)

$$A' = e\Theta, \quad A'' = \left(e + c' - \frac{ce'}{f'}\right)\Theta.$$

From this it will be easily understood in what manner the intermediate lens may be employed to increase the field. When used for this purpose it is called a *field-glass*, and is generally a lens of small power with an aperture somewhat large; and is placed between the object-glass and its principal focus.

(349.) But the compound eye-glass serves other more important purposes than that of simply increasing the field: it tends to correct the errors of form and colour.

We have already shown in what manner a compound object-glass may be constructed so as to produce an image free from all such errors: and at first sight it might appear that the errors of the eye-glass might be corrected upon the same principles; or, more simply still, that we might dispense with such separate corrections, and merely adjust the object-glass and eye-glass in such a manner as to correct each other. This, however, is impossible; for the errors of the latter, though arising from the same causes, assume a very different form, and demand a different mode of correction. To understand this diversity, it is merely necessary to observe that the axes of the several pencils incident upon the object-glass all pass through its centre, and generally at very small obliquity; while all, but those proceeding from the very middle of the field, intersect the eye-glass *excentrically*, and with considerable *obliquity*. Thus the errors of the eye-glass are much more complicated than those of the object-glass, and their correction attended with more difficulty.

The first effect of spherical aberration in the eye-glasses of telescopes is a *distortion* of the object. The axes of the extreme pencils proceeding from the centre of the object-glass will, by the aberration of the eye-glass, meet the axis of the telescope at a nearer point than those of the central pencils. The ratio of the visual angles, therefore, will be greater in the extreme parts of the field than at its central parts; these parts, accordingly, will be unduly enlarged, and the object will appear distorted.

A second effect of obliquity is *indistinctness*. The image of a distant object formed in the focus of the object-glass is, we have seen, nearly a spherical surface having the centre of the object-glass as its centre. It is therefore convex towards the eye-glass; instead of being concave towards it, as it should be in order that all its parts may be at the proper distance for distinct vision. Hence, when the middle of the image is at the proper distance from the eye-glass, the extreme parts will be too distant. If, therefore, the centre of the field be seen distinctly, the margin will be indistinct; and if, in order to have a distinct view of the latter, the eye-glass be pushed in nearer to the image, the centre of the field is rendered indistinct. This defect is very considerable where the field of view is large.

But further: the rays of a small pencil diverging from a point, and incident upon a lens *excentrically*, are not brought by refraction anywhere to a point; the rays in the plane, passing through the axis of the pencil and the axis of the lens, converging, in general, more rapidly than those in the plane passing through the axis of the pencil and perpendicular to the former. Hence, it will be seen, all the rays of the pencil converge to *two right lines;* one of which is in the plane passing through the axis of the pencil and the axis of the lens, and the other in the plane passing through the axis of the pencil and perpendicular to the former. And accordingly the image of a point, formed by such a pencil on the retina, is in general an *ellipse;* becoming however sometimes a *circle,* and sometimes a *right line.*

(350.) Such are the defects of eye-glasses dependent on their form. Without entering into details connected with this subject, it will be easily understood, in general, that the errors of distortion and indistinctness will be diminished by diminishing the aberration of the extreme pencils; and that the forms of the lenses being given, this effect will be produced by increasing their number, and thus dividing the refraction. The resulting aberration will be least, *cæt. par.*, when the whole refraction, or bending of the ray, is equally divided among the lenses. The condition of equal refraction is easily obtained; for θ being the angle made by the axis of the pencil with the axis of the telescope at the centre of the object-glass, θ', θ'', θ''', &c. $\theta^{(n)}$

the angles which it makes with the same after refraction by the several eye-glasses, the condition of equal refraction is

$$\theta' - \theta = \theta'' - \theta' = \theta''' - \theta'' = \&\text{c.} = \theta^{(n)} - \theta^{(n-1)}.$$

Hence, if θ, the original inclination of the ray to the axis, be neglected as small in comparison to the rest, we have

$$\theta'' = 2\theta', \quad \theta''' = 3\theta', \ \&\text{c.} \quad \theta^{(n)} = n\theta'.$$

Whence, dividing each equation by the preceding, there is

$$\frac{\theta''}{\theta'} = \frac{2}{1}, \quad \frac{\theta'''}{\theta''} = \frac{3}{2}, \ \&\text{c.} \quad \frac{\theta^{(n)}}{\theta^{(n-1)}} = \frac{n}{n-1}.$$

And, finally, substituting for $\frac{\theta''}{\theta'}$, $\frac{\theta'''}{\theta''}$, &c. their values,

$$\frac{c''}{d''}\left(=1-\frac{c''}{f''}\right), \quad \frac{c'''}{d'''}\left(=1-\frac{c'''}{f'''}\right), \ \&\text{c. we obtain}$$

$$c'' = -f'', \quad c''' = -\tfrac{1}{2}f''', \ \&\text{c.} \quad c^{(n)} = -\frac{1}{n-1}f^{(n)}.$$

These results, it is true, are but rough approximations, inasmuch as we have substituted the ratio of the angles themselves for that of their tangents; but they are sufficiently near the truth for our present purposes.

In the case of two eye-glasses, which we have been considering above,

$$c'' = d' + e' = -f''.$$

But $d' = -f'$, nearly, since $c'(=e)$ is very great in comparison with f', the focal length of the first eye-glass; we have therefore

$$e' = f' - f''.$$

That is, the interval between the two eye-glasses must be equal to the difference of their focal lengths. Such is the construction of the Huyghenian eye-piece.

(351.) The conditions just mentioned relate only to the *powers* and *positions* of the lenses, and are independent of their particular *forms*. It is evident, however, that the aberration will depend greatly on the form; and we have already investigated (174.) the form of a lens, whose conjugate focal distances are determined by the office which it has to perform, in which the aberration is a *minimum*. But the different effects of spherical aberration mentioned above require in general different, and sometimes opposite forms, for their correction; so that the removal or diminution of one defect frequently increases another. Thus the forms which are best for the correction of distortion may tend to increase the indistinctness of the marginal parts of the field, and *vice versâ;* and the artist is compelled to sacrifice the perfection of the instrument in one respect, in order to improve it in another which may be of more importance in the particular use to which it is to be applied. This is a subject, however, troublesome in theory, and little attended to in practice. The lenses generally employed are the plano-convex and the equi-convex lenses; and the general rule among artists is to dispose them in such a manner that the incident and emergent rays may be, as nearly as possible, equally inclined to the 1st and 2d surfaces*.

(352.) It remains still to consider the most important source of error of the eye-glasses, namely, that arising from *chromatic aberration.*

It has been already shown that the effect of the chromatic dispersion of a single eye-glass, upon a central pencil, may be disregarded as inconsiderable compared with that produced by a single object-glass (331.). It is not so, however, when the pencil is incident upon this lens *excentrically.* In this case, it is evident, the edge of the lens will act as a prism, and any single white ray of the pencil will be dispersed into its coloured elements. These different species of simple light, accordingly, intersecting the axis of the lens at different angles, the edges of the object from which they proceed will appear bordered with *coloured fringes.* This confusion will increase with the visual angle, and towards the margin of a wide field is of such magnitude as altogether to mar the appearance of the object.

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* The reader who wishes to enter further into this subject will find abundant information in professor Airy's paper *On the Spherical Aberration of the Eye-pieces of Telescopes,* published in the Transactions of the Cambridge Philosophical Society.

This error will be corrected, if all the simple rays of the emergent pencil be rendered parallel, and therefore fit to be brought to a focus on the retina. But the rays of each *homogeneous* pencil being parallel by the construction of the instrument, or nearly so, we have only to make the directions of the several component pencils, or their axes, parallel, and the thing is done. Hence, $\theta^{(n)}$ being the angle made by the axis of the emergent pencil with the common axis of the lenses, $\theta^{(n)}$ or tan.$\theta^{(n)}$ must be the same for each of the different species of simple light; and therefore its variation with respect to the refractive index must be nothing. But

$$\tan.\,\theta^{(n)} = \varrho \tan.\,\theta\,;$$

in which θ, the angle contained by the axis of the pencil with the axis of the lenses at the centre of the object-glass, is constant; and ϱ is a function of the focal lengths and intervals of the lenses, whose value has been already assigned (319.). Hence the *condition of achromatism* is

$$\Delta\varrho = 0\,;$$

$\Delta\varrho$ denoting the chromatic difference of ϱ, or the variation that quantity taken relatively to the refractive index.

(353.) In the case of two eye-glasses, we have (347.)

$$\varrho = 1 - \frac{e}{f'} - \frac{e+e'}{f''} + \frac{ee'}{f'f''}.$$

Making the difference therefore equal to nothing, and observing that $\Delta\left(\frac{1}{f}\right) = \frac{\pi}{f}$ (256.), π denoting the dispersive power, we have

$$-\frac{\pi' e}{f'} - \frac{\pi''(e+e')}{f''} + \frac{(\pi'+\pi'')ee'}{f'f''} = 0.$$

Or, multiplying by $\frac{f'f''}{e}$,

$$(\pi' + \pi'')e' = \pi' f'' + \pi'' f'\left(1 + \frac{e'}{e}\right).$$

From which we have the interval between two eye-glasses, whose focal lengths are given, when the compound is achromatic.

When the distance between the object-glass and the first eye-glass is very large (as it generally is) in comparison with the other intervals, the term $\frac{e'}{e}$ may be neglected as inconsiderable, and we have

$$e' = \frac{\pi' f'' + \pi'' f'}{\pi' + \pi''}.$$

And, finally, when the two lenses are of the same material, $\pi' = \pi''$; and the value of e' becomes

$$e' = \tfrac{1}{2}(f' + f'').$$

That is, the interval between the two lenses is an arithmetical mean between their focal lengths.

(354.) If we combine this condition with that of *equal refraction* given above, namely, $e' = f' - f''$, we find

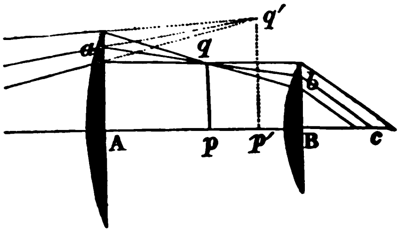
$$f' = 3f'', \quad e' = 2f''.$$

That is, the focal length of the *field-glass* must be triple that of the *eye-glass*, and the interval between them double of the same. The position of the field-glass with respect to the object-glass is determined by the equation of condition (346.): for, since $e' - f'' = f'' = \frac{1}{3}f'$, if we substitute in that equation, we find

$$e - f = -\tfrac{1}{2}f'.$$

That is, the field-glass must be placed *between* the object-glass and its principal focus, at a distance equal to half its own focal length from the latter.

Such is the construction of the Huyghenian eye-piece. It is represented in the annexed figure, in which A and B are the field-glass and the eye-glass respectively, both of them being *plano-convex* lenses with their plane sides next the eye. The rays, which by the refraction of the object-glass converge to the image $p'q'$ in its focus, are by the refraction of the field-glass made to converge to the image pq, which is in the focus of the eye-glass. It will be easily seen from the preceding formulæ that $Ap = \frac{1}{2}AB$, and $Ap' = \frac{3}{4}AB$.



(355.) This construction cannot be employed in telescopes connected with graduated instruments; for there will be a distortion of the second image produced by the field-glass, and therefore equal divisions of the *micrometer* will not correspond to equal angles. In the telescopes of all graduated instruments, therefore, the field-glass must be placed *beyond* the principal focus of the object-glass; in which arrangement, though the image there is distorted by the field-glass, yet, the micrometer wires being equally distorted, no error will result in the measurement.

In the *common astronomical eye-piece* the two lenses are of equal focal lengths, and therefore the condition of achromatism requires that the interval between them should be equal to the focal length of either. But in this arrangement, the field-glass being exactly in the focus of the eye-glass, any dust which might happen to lie upon it, or any flaw in the glass itself, would be magnified by the eye-glass and confuse the vision. The interval of the lenses therefore is made a little less than the focal length of either; and thus, though the condition of achromatism is not satisfied, yet the departure from it will not be considerable.

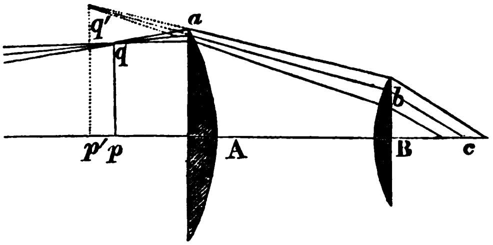
In the common construction of this eye-piece, which was invented by Ramsden, the lenses are *plano-convex* of equal curvatures, with their convexities turned towards each other; and the interval between them is *two-thirds* of the focal length of either. That is

$$f'' = f', \qquad d = \tfrac{2}{3}f'.$$

And, if we substitute these values in the equation of condition (346.), we find

$$e - f = \tfrac{1}{4} f'.$$

That is, the field-glass is placed beyond the focus of the object-glass at a distance equal to one-fourth of its own focal length. This eye-piece is represented in the annexed figure, in which



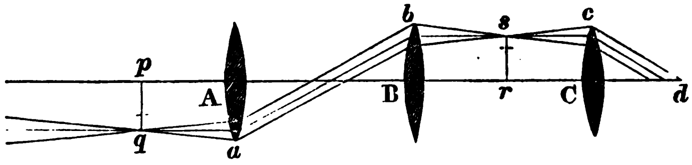
A and B represent the field-glass and eye-glass respectively, pq and $p'q'$ the images from which the rays diverge before and after refraction by the former. It is evident from the preceding results that Ap and Ap', the distances of these images from the field-glass, are equal to *one-fourth* and *one-third* of the focal length of that lens, respectively. The indistinctness arising from spherical aberration in this eye-piece is much less than in any of the other ordinary constructions.

(356.) The inversion of the object, which takes place in the different modifications of the astronomical telescope, is there of little consequence. In telescopes intended for *terrestrial* observations, however, it is absolutely necessary that the objects should be represented in their natural position; and we shall accordingly proceed, in the next place, to the consideration of eye-glasses by which this end is attained.

The erection of the object, it has been already shown, may be effected by means of the double eye-glass which we have been last considering; and it is only necessary for this purpose that the distance of the field-glass from the object-glass should be greater than the sum of their focal lengths. In this case, it is evident from the equation of condition (346.), the distance between the field-glass and eye-glass must likewise exceed the sum of their focal lengths; and accordingly the combination errs widely of the condition of achromatism (353.). This arrangement therefore is never employed.

The *common terrestrial telescope*, invented by F. Rheita, consists of three eye-glasses, the interval between the first of which and the object-glass is equal to the sum of their focal lengths, and the interval between the second and third also equal to the sum of their focal lengths; or, in other words, it is a common astronomical telescope, with two additional glasses placed at an interval equal to the sum of their focal lengths. It is obvious that, in this arrangement, the rays of each pencil are parallel between the first and second eye-glass; and that two images are formed, one at the common focus of the object-glass and first eye-glass, and another at the common focus of the second and third eye-glasses.

As this eye-piece is usually constructed, the apertures and focal lengths of the three lenses composing it are equal. It is represented in the adjoining figure, in which A, B, and C are the three eye-glasses, *qabcd* a pencil of rays proceeding from any point of the object, and refracted by the object-glass. This pencil is brought to a focus at *q* and *s*, where two images of the object *pq* and *rs* are formed, the former in the common focus of the object-glass and the lens A, and the latter in the common focus of the lenses B and C. The rays of this pencil are parallel after refraction by the lenses A and C respectively.



(357.) Without entering further at present into the examination of this particular form, let us inquire generally the construction of the three-glass eye-piece, the lenses composing it being all supposed to be convex.

The condition of distinct vision (318.) is contained in the equations

$$b = -f, \quad \frac{1}{b'} = \frac{1}{a'} - \frac{1}{f'}, \quad \frac{1}{b''} = \frac{1}{a''} - \frac{1}{f''}, \quad a''' = f''';$$

$$a' = b + e. \quad a'' = b' + e', \quad a''' = b'' + e''.$$

Now, from the extremes of the first series of equations combined with the extremes of the second, we have

$$a' = e - f, \quad b'' = f''' - e'';$$

and, these values being substituted in the second and third of the first series, the preceding equations are reduced to the three following,

$$\frac{1}{b'} = \frac{1}{e - f} - \frac{1}{f'}, \quad \frac{1}{a''} = \frac{1}{f''' - e''} + \frac{1}{f''}, \quad a'' = b' + e'.$$

And, finally, if we eliminate by substituting in the third of these equations the values of a'' and b' derived from the other two,

$$\frac{f'(f - e)}{f + f' - e} + \frac{f''(f''' - e'')}{f'' + f''' - e''} = e'.$$

When $e = f + f'$, it is evident that $e'' = f'' + f'''$, whatever be the value of e' the 2d interval; as in the common day telescope which has been just described.

(358.) The ratio of the visual angles in this combination is

$$\varrho = \frac{c'c''c'''}{d'd''d'''}.$$

If we eliminate $\frac{c'''}{d'''}$, $\frac{c''}{d''}$, $\frac{c'}{d'}$, from this expression, by means of the equations (319.), as we have already done (347.) in the case of two lenses, we find

$$\varrho = \left(1 - \frac{e}{f'}\right)\left(1 - \frac{e'}{f''} - \frac{e'}{f'''} - \frac{e''}{f'''} + \frac{e'e''}{f''f'''}\right)$$
$$- e\left(\frac{1}{f''} + \frac{1}{f'''} - \frac{e''}{f''f'''}\right);$$

or, multiplying by $f'f''f'''$,

$$\varrho . f'f''f''' = (f' - e)(f''f''' - e'f'' - e'f''' - e''f'' + e'e'')$$
$$- ef'(f'' + f''' - e'').$$

If we eliminate e from this expression, by means of the equation of the preceding article, the result will express the ratio of the visual angles when the combination fulfils the condition of distinct vision, or the magnifying power of the telescope. Performing this operation we find

$$\varrho \cdot f'f''f''' =$$

$$-f\left\{(f'f''+f'f'''+f''f''')-(e'f''+e'f'''+e''f'+e''f'')+e'e''\right\}.$$

When $e''=f''+f'''$, as in the common day telescope, the second member of this equation is reduced to ff''^2, and the magnifying power of the instrument therefore is

$$\varrho = \frac{ff''}{f'f'''}.$$

Whence it appears that the magnifying power of the common astronomical telescope is altered by the two additional eye-glasses, in this construction, in the ratio of the focal lengths of these glasses; and that when the focal lengths of the three eye-glasses are equal, as is commonly the case, the magnifying power is unaltered.

(359.) The apertures of the eye-glasses are determined by the equations

$$\text{A}' = e\Theta, \quad \text{A}'' = \left(e + e' - \frac{ee}{f'}\right)\Theta,$$

$$\text{A}''' = \left(e + e' + e'' - \frac{e(e'+e'')}{f'} - \frac{e''(e+e')}{f''} + \frac{ee'e''}{f'f''}\right)\Theta.$$

Since e is very considerable, with respect to e' or e'', we may without much error neglect all the terms in which it is not involved; and the preceding expressions thus become

$$\text{A}'=e\Theta, \quad \text{A}''=e\Theta\left(1-\frac{e'}{f'}\right), \text{A}'''=e\Theta\left(1-\frac{e'}{f'}-\frac{e''}{f'}-\frac{e''}{f''}+\frac{e'e''}{f'f''}\right).$$

In the common day-telescope, in which the three eye-glasses are of equal focal lengths, $f''=f'$, and $e''=2f'$, and the expression of the aperture of the third lens becomes

$$\text{A}''' = e\Theta\left(\frac{e'}{f'} - 3\right).$$

The magnitude of the apertures of the second and third lenses, accordingly, will depend upon the relation which the second interval bears to the common focal length. When $e' = 2f'$, we have

$$\mathrm{A}'' = - e\Theta, \qquad \mathrm{A}''' = - e\Theta\,{}^{*};$$

from which it appears that the apertures of the three lenses are equal, when the interval between the second and third lenses is double the common focal length. Such we find, accordingly, is the usual construction.

(360.) We now proceed to inquire the condition of achromatism in the triple eye-piece.

The interval between the object-glass and the first eye-glass being, in general, very considerable in relation to the focal length of the latter, we may, without much error, neglect f' in comparison with e in the general expression of ϱ (358.); and, dividing that equation by $f'f''f'''$, we find

$$\varrho = - e\left\{\frac{1}{f'}+\frac{1}{f''}+\frac{1}{f'''}-\left(\frac{e'}{f'f''}+\frac{e'}{f'f'''}+\frac{e''}{f'f'''}+\frac{e''}{f''f'''}\right)+\frac{e'e''}{f'f''f'''}\right\}.$$

Now taking the chromatic difference of this quantity, and confining our consideration to the case in which the lenses are all of the same material, we find

$$\Delta\varrho = - e\varpi\left\{\frac{1}{f'}+\frac{1}{f''}+\frac{1}{f'''}-\left(\frac{2e'}{f'f''}+\frac{2e'}{f'f'''}+\frac{2e''}{f'f'''}+\frac{2e''}{f''f'''}\right)+\frac{3e'e''}{f'f''f'''}\right\};$$

ϖ denoting the dispersive power of the substance of which the lenses are composed. And finally, equating this to nothing, and multiplying the resulting equation by $f'f''f'''$, we have

$$3e'e'' - 2\left[e'(f''+f''') + e''(f'+f'')\right] + f'f'' + f'f''' + f''f''' = 0;$$

the condition of achromatism of the triple eye-piece, the lenses composing it being all of the same material.

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[*] The negative sign here denotes that the axis of the extreme pencil meets the apertures of the second and third lenses at the side of the common axis opposite to that at which it meets the first.

In the Rheita eye-piece, $e'' = f'' + f'''$; and, substituting, the interval between the first and second eye-glass, when the compound is achromatic, will be

$$e' = f' + f'' + \frac{f''^2}{f'' + f'''}.$$

And when the focal lengths of the three lenses are equal, or $f' = f'' = f'''$,

$$e' = \tfrac{5}{2} f'.$$

Accordingly the common construction of this eye-piece, in which $e' = 2f'$, is not achromatic; and if it be rendered so, by increasing this interval to the magnitude here required, a loss of field will ensue, unless the apertures of the lenses be adjusted to the new distances, and therefore cease to be equal.

(361.) A more important objection to the construction last mentioned is, that the bending of the ray cannot be equally divided among the three refractions, when the condition of achromatism is fulfilled. In order that the bending may be equal at the first and second eye-glass, the axis of the extreme pencil must emerge from the latter parallel to the axis of the lenses; since, neglecting the small original inclination of this axis to the axis of the lenses, it may be supposed parallel at its incidence on the first eye-glass. Hence this axis must intersect the axis of the lenses at the common focus of the first and second eye-glasses, and therefore the interval between them must be equal to the sum of their focal lengths, or

$$e' = f' + f'';$$

which is necessarily less than the value of the interval which will render the Rheita eye-piece achromatic. Substituting this value of e' in the general condition of achromatism given in the preceding article, we find

$$e'' = f'' + f''' + \frac{f''^2}{f' + f''}.$$

That is, the interval between the second and third lenses must exceed the sum of their focal lengths by a quantity which is a third proportional to the interval between the first and second lenses and the focal length of the latter. Such is the determination of Boscovich, the first writer who has given any satisfactory information on the subject of the achromatism of eye-pieces. Boscovich recommends that the focal lengths of the first and second eye-glasses of the triple eye-piece should be equal, and double that of the third, or that

$$f' = f'' = 2f'''.$$

In which case

$$e' = e'' = 2f';$$

and substituting these values in the equation (357.),

$$e - f = \tfrac{1}{2}f'.$$

We learn accordingly that the common day-telescope may be rendered achromatic, the intervals between the eye-glasses remaining unaltered, by substituting for the third eye-glass one of half the focal length, and pushing in the eye-piece until the distance of the first eye-glass from the image formed by the object-glass is equal to one-half its focal length. The field is necessarily diminished in this construction, because the third eye-glass, having double its former power, will not sustain so great an aperture as before. We may, however, restore this aperture, and therefore the field of view, to its former dimensions, by substituting for the single eye-glass of double power two lenses in contact, the focal lengths of each being equal to that of the first or second eye-glass.

(362.) In this construction, however, the bending of the extreme pencil is not the same at the second and third lenses. This pencil being parallel to the axis between these lenses, it is evident that their focal lengths must be equal in order that the axis of the pencil may be equally refracted by them. If, therefore, we take

$$f' = f'' = f''',$$

in the values of e' and e'' given in the preceding article, we find

$$e' = 2f', \quad e'' = \tfrac{5}{2}f'.$$

And substituting these values in the equation (357.), the first interval will be

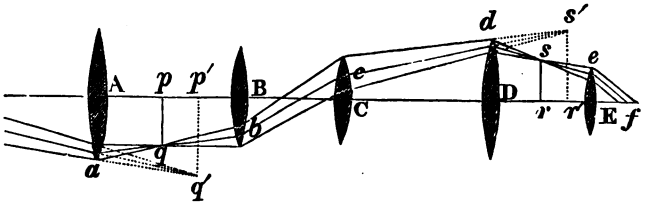
$$e = f + \tfrac{1}{2}f'.$$

Finally, if these values of e', e'', and f'', be substituted in the equations (359.), we find

$$\text{A}'' = \text{A}''' = -e\Theta = -\text{A}'.$$

The apertures and focal lengths of the three lenses, therefore, are equal in this construction; which differs accordingly from the common day-telescope only in the intervals between the lenses. It appears then that the most advantageous method of rendering the common day-telescope achromatic is to increase the distance between the second and third lenses by one-half their common focal length, and to diminish the distance of the first lens from the object-glass by the same quantity. The latter adjustment is effected by simply pushing in the eye-tube in which the eye-glasses are fixed. The magnifying power will be the same as before, that is, the same as if one eye-glass only had been employed; as will appear by substituting the preceding values in the general value of ϱ (358.).

(363.) In the three-glass eye-piece, the refraction at each lens is considerable, since the extreme pencil is made to intersect the axis twice. It will be advantageously modified therefore by adding to the number of lenses, and thus diminishing the refraction of each. It occurred to Dollond to substitute the Huyghenian double eye-glass for the first and third lenses of the triple eye-piece. In this manner an eye-piece of *five* glasses is produced, in which the field is increased, and the bending at each lens considerably diminished. It is represented in the adjoining figure, in which A, B, C, D, E represent the several lenses taken in their order; *abcdef* the course of the extreme pencil, intersecting the axis of the telescope between the second and third lenses, B and C, and a second time after refraction by the last lens E. The rays of this pencil are brought to a focus at *g* and *s*, and thus two



images, *pq* and *rs*, are formed, the first of which is between the first and second lenses, and the second between the fourth and fifth. An image of the object-glass is formed at each of the points in which the axis of the extreme pencil intersects the axis of the lenses; and the whole of the light passes through a small circle there. It is usual to place at each of these points a plate perforated with an aperture equal to the magnitude of the image formed there, in order to stop erratic light.

This eye-piece was afterwards laid aside by Dollond for one with *four* glasses, which is now the most approved form of the eye-piece for terrestrial telescopes. Its construction will be understood from the preceding figure by simply supposing the first lens A to be removed. Thus the first image is formed *before* the first eye-glass, and the second between the third and fourth; and the axis of the extreme pencil crosses the axis of the lenses between the first and second. The confusion arising from spherical aberration is considerably less in this than in the triple eye-piece; and it is rendered achromatic by fulfilling a condition similar to that which we have already investigated in the cases of the double and the triple eye-piece.

V.

Of the Compound Microscope.

(364.) It has been already stated that the compound microscope is, essentially, nothing but a common astronomical telescope adapted to near objects; and that this adaptation requires that the object-glass should be of very short focal length, and therefore of very small aperture. The microscopic object being placed at a distance from this lens a little greater than its focal length, an inverted and magnified image will be formed at the other side of the lens, and at a distance which is considerable as compared with that of the object. This image is viewed, as in the common astronomical telescope, through an eye-glass placed at a distance from it equal to its own focal length, and thus a second amplification will take place with all the circumstances necessary to distinct vision.

The condition of distinct vision, or of the parallelism of the emergent rays, is contained in the equations

$$\frac{1}{b} = a - \frac{1}{f}, \qquad a' = b + e, \qquad a' = f';$$

a, b, a', e, f, and f', denoting as before (318.). Or, eliminating b and a',

$$\frac{1}{a} = \frac{1}{f} + \frac{1}{f' - e}.$$

Such is the relation which must subsist among the distance of the object, the interval of the lenses, and their focal lengths, in order that the emergent rays may be suited to distinct vision. The interval of the lenses is usually fixed in this instrument, and the adjustment to distinct vision is effected by varying a, the distance of the object from the object-glass; and for this purpose the object is placed on a stage, whose distance from the object-glass is altered by means of a rack and pinion. Sometimes the stage is fixed and the body of the instrument moveable; but the principle of the adjustment is the same in the two cases.

For near-sighted persons, whose eyes require diverging rays, the value of a must be less than that assigned by the preceding equation: for far-sighted persons it must be greater.

(365.) To determine the magnifying power of this instrument, we have

$$\varrho = \frac{d}{d'} = 1 - \frac{e}{f'};$$

c' and d' denoting as before (319.). But, if M denote the magnifying power, and λ the least distance of distinct vision,

$$\mathrm{M} = \frac{\lambda}{a}\varrho;$$

and, substituting for $\frac{1}{a}$, and for ϱ, their values obtained above, we find

$$\mathrm{M} = \lambda\left(\frac{1}{f} + \frac{1}{f'} - \frac{e}{ff'}\right).$$

Such is the value of the magnifying power of the compound microscope, when adapted to eyes which require parallel rays. If the instrument be adjusted to the vision of a near-sighted person, it is evident from the preceding article that the magnifying power will be greater than that here assigned; while, on the other hand, it will be less when the instrument is adjusted to the vision of one who is far-sighted.

The magnifying power of the compound microscope is readily ascertained in practice. It is only necessary for this purpose to measure the distance of the object from the object-glass, when seen distinctly, and the diameter of the image of the object-glass, which is formed beyond the eye-glass at the intersection of the axes of the extreme pencils with the common axis of the lenses. The ratio of the diameter of the object-glass to that of this image will give the value of ϱ (321.); and this ratio, multiplied by the ratio of the least distance of distinct vision to the distance of the object from the object-glass, is equal to the magnifying power.

(366.) The quantity of light in the image of an object, seen through the compound microscope, is equal to $\left(\frac{\mathrm{A}\lambda}{a\alpha}\right)^2$, that of the image of the same object seen by the naked eye, at the least distance of distinct vision, being unity. The *penetrating power* of the instrument is proportional to the absolute quantity of light in the image of an object seen through it, and therefore varies as $\left(\frac{\mathrm{A}}{a}\right)^2$, or in the duplicate ratio of the angle which the diameter of the object-glass subtends at the object.

The brightness of the image of an object seen through the compound microscope is equal to $\left(\frac{\text{A}}{\text{M}a}\right)^2$, that of the same object seen by the naked eye being unity. Hence the brightness in this instrument, as in the telescope, varies in the duplicate ratio of the aperture of the object-glass divided by the magnifying power.

(367). The confusion arising from the spherical aberration of the lenses varies in the duplicate ratio of the angle of aberration, which is equal to

$$-\tfrac{1}{2}\frac{f'}{b}\left\{\left(\frac{b}{f'}\right)^2 x + \left(\frac{f'}{b}\right)^2 x'\right\}\text{A}^3;$$

as appears by substituting f' for a' in the result of (330). Or, if we make $x = \frac{\text{L}}{f^3}$, $x' = \frac{\text{L}'}{f'^3}$, the preceding value becomes

$$-\tfrac{1}{2}b\left\{\frac{\text{L}}{f'f^3} + \frac{\text{L}'}{b^4}\right\}\text{A}^3.$$

The values of L and L′ in this formula are given by the equation of (173.); the latter accurately, and the former very nearly, since the object is nearly in the focus of the object-glass. If the second term of the quantity within the brackets be neglected, on account of the magnitude of b, compared with that of f or f'; and if for b its value $f' - e$ be substituted, the expression of the angle of aberration is reduced to

$$\tfrac{1}{2}\text{L}\left(\frac{e}{f'} - 1\right)\left(\frac{\text{A}}{f}\right)^3.$$

The magnitude of the coefficient L depends upon the material of which the object lens is composed, and upon its form. It is evident from what has been said (315.) that the higher the refractive power of the substance of which this lens is composed, the less, *cæt. par.*, will be the value of L, and, consequently, the less the confusion. The form of the lens for which the value of L is a *minimum* is determined by the equations of

(174.); and it is evident from what has been said that, if the lens be of glass, the aberration of the lens of *best form* will not differ much from that of a *plano-convex*, having its *plane* surface turned towards the object. Such, accordingly, is the usual form of the object-glasses of the compound microscope.

(368.) The confusion arising from chromatic aberration varies in the duplicate ratio of the angle which the diameter of the circle of chromatic aberration in the last image subtends at the eye, or at the eye-glass qp. If we make $a' = f'$ in the general result of (331.), the value of this angle is found to be

$$-b\mathrm{A}\left\{\frac{\pi}{ff'} + \frac{\pi'}{b^2}\right\}.$$

If we neglect the second term of the quantity within the brackets in comparison with the first, on account of the magnitude of b compared with that of f or f', and substitute in the result for b its value $f' - e$, the approximate value of the angle of dispersion is

$$\pi\left(\frac{e}{f'} - 1\right)\frac{\mathrm{A}}{f}.$$

This angle varies therefore, *cæt. par.*, as the aperture of the object-glass divided by its focal length; and the confusion arising from chromatic dispersion varies as the square of that quantity.

(369.) On a comparison of the two preceding results, it appears that the confusion of vision in the compound microscope, whether arising from the chromatic dispersion of the object-glass, or from its spherical form, will depend, *cæt. par.*, on the ratio which the aperture of that lens bears to its focal length, and upon the ratio which the interval between it and the eye-glass bears to the focal length of the latter.

The errors both of form and colour, however, may be removed, as in the achromatic telescope, by employing a *compound object-glass* consisting of two or more lenses in contact of different refractive and dispersive powers. The condition of achromatism in such a lens will evidently be the same as in the compound object-glass of the achromatic telescope; since, for any system of lenses in contact, the condition of achromatism is altogether independent of the distance of the radiant point. Hence, if the object-glass of the compound microscope consists of two lenses in contact, their focal lengths must be in the ratio of the dispersive powers of the substances of which they are composed.

The determination of the forms of two lenses, which constitute an aplanatic combination when placed in contact, is given by the equations of (185-6.), the development of which we shall leave to the reader. The calculation is more difficult than in the case of the object-glass of a telescope, inasmuch as the equations necessarily involve the distance of the radiant point, which in the latter case is infinite.

The first application of the achromatic object-glass to the compound microscope was made a few years since by Mr. Tully, at the suggestion of Dr. Goring. The object-glasses made by this artist are *triple*, and are composed of a concave lens of *flint-glass* between two convex lenses, one of which is of *crown-glass* and the other of *Dutch plate.* These object-glasses sustain an aperture equal to half their focal lengths, which are from 0.2 to 1.0 inch; whereas the aperture of the single object-glass cannot be much greater than the one-eighth of its focal distance. By this increase of aperture the quantity of light, and therefore the penetrating power of the instrument, is much augmented.

(370.) One of the chief advantages of the compound microscope, as compared with the single microscope of equivalent power, consists in its extent of field. If m denote the linear magnitude of the field of view in this instrument, and A' the aperture of the eye-glass, it is evident from similar triangles that

$$m = \frac{a}{e}\text{A}';$$

in which the value of a is given by the condition of distinct vision. Such is the extent of the field as limited by the extreme axes transmitted by the eye-glass. The entire visible field is somewhat greater, and the bright part of the field somewhat less, than this quantity; and it is evident that their magnitudes will be ascertained by a method precisely similar to that employed in the case of the astronomical telescope (336-7.). On account of the smallness of the aperture of the object-glass, however, and the magnitude of the interval between the lenses as compared with the focal length of either, these will differ little from the mean field above determined.

In order to increase the field, an additional lens is generally introduced, which is therefore called the *field-glass.* The theory of the instrument in this form will be sufficiently understood from what has been said, in the preceding section of this chapter, on the subject of the double eye-pieces of telescopes.

CHAPTER IV.

OF VISION BY MIRRORS AND LENSES COMBINED.

I.

Of Sir William Herschel's and of the Newtonian Telescope.

(371.) If, instead of a convex lens, a concave reflecting speculum be employed to receive the rays proceeding from a distant object, an image will be formed at its focus, which, if the aperture of the speculum be sufficiently large, may be viewed directly through an eye-glass placed at the distance of its own focal length from the image, as in the common astronomical or Galilean telescopes.

Such is the principle of the reflecting telescope in its simplest form, and was that adopted by Sir William Herschel in the construction of his celebrated instruments. In order that the head of the spectator may intercept as little light as possible, the axis of the speculum is slightly inclined to the direction of the incident rays, and thus the image thrown near the edge of the tube, where it is viewed through the eye-glass by the spectator having his back towards the object. This method of observation is called by Sir William Herschel the *front view*.

It is evident that the obliquity of the incident pencil in this construction must produce a slight distortion of the image; but the errors thence arising are scarcely appreciable in the very large instruments, to which alone this construction can be applied.

(372.) The angle under which the object is seen with this instrument is equal to $\frac{m}{f}$, m denoting the linear magnitude of the image formed by the concave speculum, and f the focal length of the eye-glass. But the angle which the object subtends to the naked eye, or at the centre of the speculum *q.p.*, is equal to that subtended by its image at the same place, with the opposite sign; or equal to $\frac{-m}{F}$, F denoting the focal length of the object-speculum. Wherefore the ratio of these angles, or the *magnifying power*, is, as in the common astronomical telescope,

$$M = -\frac{F}{f}.$$

It is evident that an object seen through this instrument will appear inverted.

(373.) The *field of view* in this telescope, determined by the axes of the extreme pencils, is equal to the angle which the aperture of the eye-glass subtends at the centre of the concave speculum, or at a distance equal to the difference of the focal lengths of the speculum and eye-glass. That is, if a denote the semi-aperture of the eye-glass, and θ half the field of view,

$$\theta = \frac{a}{F - f};$$

which, since the focal length of the eye-glass in these instruments is very small in comparison with that of the object-speculum, is very nearly equal to the angle which the aperture of the eye-glass subtends at the vertex of the speculum.

The *entire visible field*, from which any rays whatever reach the eye, is found by joining the corresponding extremities of the apertures of the eye-glass and object speculum; the intercepted portion of the perpendicular erected at their common focus will be the linear magnitude of the image which is illuminated by any rays whatever proceeding from the object, and the angle which it subtends at the centre of the object-speculum will be the extreme field. It is evident that the determination is precisely the same as in the case of the common astronomical telescope, so that, adopting the result of (336.), we have

$$\Theta' = \frac{A\frac{f}{F} + a}{F + f};$$

A denoting the semi-aperture of the object-speculum, and Θ' half the extreme field. Or, neglecting f in comparison with F, and substituting M for $\frac{F}{f}$,

$$\Theta' = \frac{1}{F}\left(\frac{A}{M} + a\right);$$

a value which, in instruments of high magnifying power, will differ little from that of the mean field given above.

On account of the small extent of field in these instruments, and the consequent difficulty of discovering the object, it is usual to attach, at the side of the tube, a small refracting telescope of low power and considerable field, whose axis is exactly parallel to that of the telescope to which it is joined. The object being found by means of this telescope, it is evident that the larger instrument will be in the desired position. Such an accessory is called a *finder*.

(374.) The celebrated instrument, erected at Slough by Sir W. Herschel, was the largest reflecting telescope ever made. The tube of this telescope is thirty-nine feet four inches in length, and four feet ten inches in diameter. It is made of rolled or sheet iron, and is put together without rivets by a kind of seaming used in the iron funnels of stoves. Great mechanical contrivance is displayed in the apparatus destined for the management and support of this vast instrument. The lower extremity of the tube rests upon a moveable point of support, and the upper or open end is attached by means of pullies to a great cross beam which crowns the framework of the whole apparatus, and by which the telescope may be set to any desired altitude. The whole instrument, framework and all, has also an azimuth motion by the manner of its support. The foundation on which it rests consists of two concentric brick walls, carefully capped with stone, the outermost of which is 42 feet in diameter and the innermost 21 feet. The framework rests upon these walls by means of twenty concentric rollers, moving on a pivot in the centre, and thus has an easy horizontal motion.

The diameter of the metallic speculum is $49\frac{1}{2}$ inches, and that of its polished surface 48 inches. The thickness, which is uniform throughout, is $3\frac{1}{2}$ inches; and its weight, when it came from the furnace, was 2118 pounds. The focus of the speculum is made, by a slight inclination of its axis, to fall within four inches of the lower part of the mouth of the tube, and projects into the air; by which arrangement the head of the spectator intercepts little of the incident light, the diameter of the tube exceeding the effective aperture of the speculum by about ten inches.

A slider, carrying a brass tube for the reception of the eye-glasses, is fixed at the mouth of the telescope, and is directed to the vertex of the speculum. The eye-glasses employed by Sir W. Herschel were small double-convex lenses, some of which are said not to exceed one-fiftieth of an inch in focal length. With eye-glasses, however, such as are more commonly employed for sidereal observations, the magnifying power of the telescope exceeds 6000.

A speaking-pipe descends from the mouth of the telescope to the bottom of the tube, and there divides into two branches; one of which communicates with the work-room, in which a person is placed to give the required movements to the instrument, and the other with the observatory, where an assistant takes down the observations. In this latter room there is a sidereal time-piece, and close beside it a polar-distance piece, which may be made to show polar-distance or declination, zenith-distance or altitude, by setting it differently. The speaking-pipe rises between the time-piece and polar-distance piece, and the assistant sits in front of them at a table, and there writes down the observations *.

This noble instrument was completed on the 28th of August, 1789, and on the same day the sixth satellite of Saturn was

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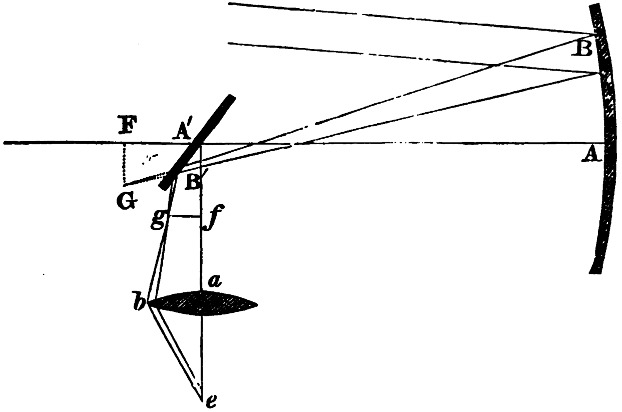
* A full account of this telescope, and of the mechanical arrangements connected with it, is given in the Transactions of the Royal Society for the year 1795.

discovered. The framework having much decayed, this telescope has been taken down, and another of the same construction, of 20 feet focal length and 18 inches diameter, erected in its place by Mr. J. F. Herschel. The largest telescope of this construction, now in this country, is that erected by Mr. Ramage at the Royal Observatory at Greenwich, in the year 1820. Its focal length is 25 feet, and the aperture of the object-speculum 15 inches. The mechanical arrangements of this instrument are such that its movements can be effected by the observer himself without an assistant.

(375.) The principle of the *front view*, as described in the preceding constructions, can only be employed, it is evident, when the aperture of the object-speculum is very considerable: to instruments of moderate dimensions it is wholly inapplicable. In the telescope proposed and constructed by Sir Isaac Newton, the rays reflected by the object-speculum are received upon a small plane speculum placed between the object-speculum and its principal focus, the plane of which is inclined at an angle of 45° to the axis of the telescope. By this the rays, which tend to form an image at the principal focus of the object-speculum, are reflected laterally; and thus an image is formed near the side of the tube, equal and similar to the former, and similarly posited with respect to the plane reflector. This image, whose plane is parallel to the axis of the telescope, is viewed through an eye-glass placed at the side of the instrument, at a distance from the image equal to its own focal length.

Instead of a plane speculum, Newton made use of a rectangular isosceles prism, through the sides of which the rays enter and emerge perpendicularly, and are reflected at the hypothenuse. The reflexion there being *total*, the loss of light will be much less than in the case of a metallic speculum.

The construction of the Newtonian telescope is represented in the adjoining figure, in which AB represents the object-speculum, A'B' the plane speculum, and *ab* the eye-glass. BB'*gbe* represents part of a pencil of rays proceeding from any point of a distant object. These rays, which by the reflexion of the object-speculum are made to converge towards G, the



corresponding point of the image, FG, which would be formed at its principal focus, are intercepted by the plane mirror, A'B', and made to converge to *g*, where an image, *fg*, will be formed equal to FG, and similarly situated with respect to A'B'. The rays then crossing at *g* are incident diverging upon the eye-glass, *ab*, placed at a distance, *af*, from the image equal to its own focal length; and finally, emerging parallel, are received by the eye at *e*, the intersection of the axis of the pencil with the axis of the lens.

(376.) If F and f denote the focal lengths of the object-speculum and eye-glass, respectively, e and e' their distances from the plane reflector, AA' and A'a, we have

$$\text{A}'\text{F} = \text{F} - e, \qquad \text{A}'f = e' - f;$$

the first image being in the principal focus of the object-speculum, and the second in that of the eye-glass. Wherefore, since A'F = A'f, there is

$$e + e' = \text{F} + f.$$

Such is the condition of distinct vision of a distant object, the eye requiring parallel rays. The position of the eye-glass with respect to the plane speculum is usually fixed, or the distance e' invariable; and the adjustment to distinct vision is effected by varying e, the distance of the plane mirror from the object-speculum. The motion of the plane mirror, and of the eye-glass which is attached to it, is effected by means of a fine screw.

The magnifying power of the Newtonian telescope is evidently determined on the same principles as that of Sir W. Herschel; and is equal to $\frac{\mathrm{F}}{f}$ the ratio of the focal lengths of the object-speculum and eye-glass.

(377.) In order that the brightness of the central part of the field may be as great as possible, the plane mirror must be of such form and dimensions as exactly to receive the whole of the principal pencil, or the cone of rays converging to the principal focus of the object-speculum. The mirror, accordingly, must be a section of that cone, formed by a plane inclined at an angle of 45° to its axis; and must therefore be an *ellipse.*

In order to determine the dimensions of this ellipse, let lines be conceived drawn from the focus F, in the preceding figure, to the extremities of the aperture of the object-speculum; and let the section of the lesser speculum, A′B′, be produced to meet them. Then, this section being made by a plane passing through the axis of the telescope and perpendicular to the plane of the speculum, it is evident that the intercepted portion of the section will be the greater axis of the ellipse. Let x and x' denote the portions of that line, as it is divided in the point A′ by the axis of the telescope; also, let A′F, the distance of the focus from the lesser speculum, be denoted by d; and the semiangle of the cone by θ, the tangent of which is equal to $\frac{\mathrm{A}}{\mathrm{F}}$. Then we have

$$x = \frac{d.\sin.\theta}{\sin.(45-\theta)} = \frac{d\sqrt{2}.\tan.\theta}{1-\tan.\theta} = \frac{\mathrm{A}d\sqrt{2}}{\mathrm{F}-\mathrm{A}};$$

$$x' = \frac{d.\sin.\theta}{\sin.(45+\theta)} = \frac{d\sqrt{2}.\tan.\theta}{1+\tan.\theta} = \frac{\mathrm{A}d\sqrt{2}}{\mathrm{F}+\mathrm{A}}.$$

Wherefore, a denoting the semiaxis major of the ellipse, we have

$$a = \tfrac{1}{2}(x + x') = \frac{\text{AF}d\sqrt{2}}{\text{F}^2 - \text{A}^2}.$$

Again: if b denote the semiaxis minor, and y the perpendicular to the greater axis erected at the point A', by the property of the ellipse, there is

$$\frac{b^2}{a^2} = \frac{y^2}{xx'}.$$

But y is evidently the radius of the circle, formed by a plane perpendicular to the axis of the cone at the point A': wherefore

$$y = d\tan.\theta = \frac{\text{A}d}{\text{F}}, \quad \text{and} \quad \frac{b^2}{a^2} = \frac{\text{F}^2 - \text{A}^2}{2\text{F}^2}.$$

And substituting for a its value already obtained,

$$b = \frac{\text{A}d}{\sqrt{\text{F}^2 - \text{A}^2}}.$$

If A^2 be neglected in comparison with F^2, on account of the smallness of the aperture of the object-speculum as compared with its focal length, the approximate values of the semiaxes will be

$$a = \frac{\text{A}d\sqrt{2}}{\text{F}}, \quad b = \frac{\text{A}d}{\text{F}};$$

which are in the ratio of $\sqrt{2}$ to 1. This latter determination is sufficiently near the truth for all practical purposes.

The quantity of light reflected by a plane mirror increases with the incidence. Consequently, if the plane speculum be inclined to the axis of the telescope at a greater angle than 45°, the quantity of light and brightness of the image will be augmented. In this modification of the Newtonian telescope, which was proposed by Dr. Brewster, the rays emerge through the side of the tube obliquely. The form of the elliptical mirror also, depending on its inclination to the axis of the cone, will differ from that assigned above; and it is evident that its greater axis will be divided more unequally by the axis of the telescope, the greater the inclination.

(378.) The extreme field of view in the Newtonian telescope is determined by connecting the adjacent extremities of the eye-glass and plane speculum by the line $\text{B}'b$ (see figure, page 362); the intercepted portion of the perpendicular, fg, raised at the point f, will be the whole extent of the image illuminated by any rays whatever proceeding from the object, and the angle which its equal FG subtends at the centre of the speculum will be the extreme field.

Let m denote the linear magnitude of this image, a the semi-aperture of the eye-glass, ab, and A' the perpendicular let fall from the adjacent extremity of the plane speculum, B', upon the line $\text{A}'a$; then it is evident that we have, on account of similar triangles,

$$\frac{a-m}{f}=\frac{m-\text{A}'}{d-\text{A}'};$$

d denoting as before the distance $\text{A}'\text{F}$ or $\text{A}'f$. Neglecting A' in comparison with d in this result, we obtain

$$m=\frac{ad+\text{A}'f}{d+f}.$$

And this quantity divided by F will measure half the visible field.

From the preceding article it appears that $\text{A}'=\text{A}\frac{d}{\text{F}}$, very nearly. Wherefore, if we substitute this value in the result last obtained, and neglect f in comparison with d in the denominator, it is reduced to

$$m=a+\frac{\text{A}}{\text{M}}.$$

Whence, Θ denoting half the visible field, there is

$$\Theta=\frac{1}{\text{F}}\left(a+\frac{\text{A}}{\text{M}}\right);$$

as in Herschel's telescope.

(379.) If a real object be supposed to occupy the place of the image fg (see figure, page 362), it is evident that the rays of each pencil, after reflexion by the plane speculum, A′B′, will fall upon the concave speculum diverging from the image FG in its principal focus; the reflected rays, consequently, emerging parallel, will be suited to the distinct vision of an eye placed any where in the axis of the instrument. But if the object be removed ever so little farther from the plane speculum, it is evident that the rays, after reflexion by the two specula, will converge to an image in the axis of the concave; and this image may be viewed through an eye-glass placed at a distance from it equal to its focal length.

Such is the principle of the *reflecting microscope*, invented by Professor Amici. In order to avoid the errors of figure, the concave mirror is an *ellipsoid* of revolution round the greater axis, whose foci are the places of the conjugate images. The microscopic object is placed a little without the tube, on a small shelf projecting from the stand of the instrument; and is illuminated by two concave mirrors, one of which is directly above the object, and has an aperture in the centre to admit the rays into the tube, and the other directly beneath it.

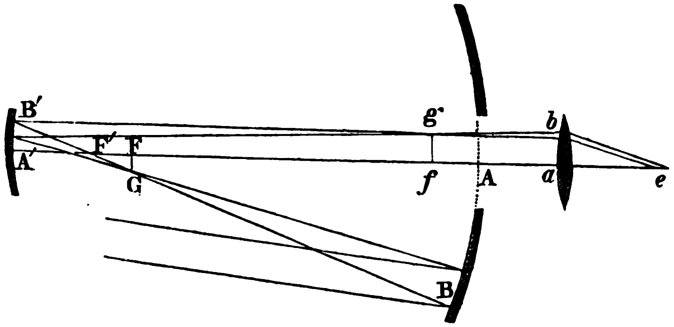
This instrument admits of a very high magnifying power, and, owing to its horizontal position, is very convenient in use. Its construction has been considerably improved by Mr. Cuthbert, principally by altering the relative dimensions of the two specula; and it is now much in use.

II.

Of the Gregorian and Cassegrainian Telescopes.

(380.) The invention of the reflecting telescope is generally ascribed to James Gregory, who described the instrument now called by his name in his *Optica promota*, published in the year 1663. The Gregorian telescope consists of two concave specula disposed along the same axis, with their concave surfaces facing each other, and at an interval a little greater than the sum of their focal lengths. In the vertex of the larger, or object-speculum, is a circular aperture, to which is attached a tube containing the eye-glass. Now, when the axis of the telescope is turned towards any distant object, an image is formed at the principal focus of the object-speculum. The rays diverging from this image are then received by the lesser speculum, which is at a distance from it a little greater than its own focal length; and by its reflexion a second image is formed near the vertex of the object-speculum, which is viewed through the eye-glass placed there, at a distance from it equal to its own focal length.

The construction of this instrument is represented in the adjoining figure; in which AB represents the object-speculum,



A′B′ the lesser speculum, and *ab* the eye-glass. BGB′*gbe* represents part of a pencil of rays proceeding from any point of a distant object. These rays are, by the reflexion of the object-speculum, made to converge to G, the corresponding point of the image, FG, formed at its principal focus F. The distance of this image from the lesser speculum, A′F, being a little greater than its focal length A′F′, the rays diverging from G, and incident upon the lesser speculum, will be reflected to *g*; where a second image, *fg*, will be formed at a distance A′*f* conjugate to A′F. This second image being in the principal focus of the eye-glass, *ab*, the rays diverging from *g*, and incident upon the eye-glass through the aperture in the object-speculum, are by its refraction made to emerge parallel; and are received by the eye at *e*, the intersection of the axis of the pencil with the axis of the instrument.

In the original construction of this telescope, as proposed by its inventor, the figure of the object-speculum was a *paraboloid*, and that of the lesser speculum an *ellipsoid* of revolution round the greater axis, whose foci are F and f, the places of the two images. In such an arrangement, it is evident, there will be no aberration of the rays proceeding from the centre of the field; and for such rays the instrument will be theoretically perfect. On account of the difficulty, however, attending the construction of mirrors of these forms, which were at first considered as essential to the instrument, this invention was for some time abandoned as practically hopeless; so that, although the Gregorian telescope was invented some years before that of Newton, the latter was the first reflecting telescope actually constructed.

For common purposes the Gregorian telescope is generally preferred to the Newtonian. Its superiority seems to arise from this, that the two specula may be so matched that the necessary irregularities in their figure shall compensate one another in their effects upon the images; whereas in the Newtonian there is nothing to counteract any defect of form of the object-speculum, and experience proves that such specula can seldom be obtained truly spherical.

(381.) Let F, F', and f, denote the focal lengths of the object-speculum, the lesser speculum, and the eye-glass respectively; e and e' the distances of the object-speculum and eye-glass from the lesser speculum, and d and d' the distances of the two images from the same: then, since the first image is in the principal focus of the object-speculum, and the second in that of the eye-glass, it is evident that

$$d = e - \text{F}, \qquad d' = e' - f.$$

But d and d', being conjugate focal distances to the lesser speculum, are connected by the equation $\frac{1}{d} + \frac{1}{d'} = \frac{1}{\text{F}'}$; and substituting in this the values of d and d', just given, there is

$$\frac{1}{e - \text{F}} + \frac{1}{e' - f} = \frac{1}{\text{F}'}.$$

Such is the equation of condition of distinct vision in the Gregorian telescope, an equation analogous to that already obtained (346.) in the case of the refracting telescope with three lenses.

The eye-glass is usually fixed in this telescope, as in the Newtonian, and the adjustment to distinct vision is effected by altering the position of the lesser speculum by means of a fine screw.

(382.) The angle under which any object is seen in the Gregorian telescope is equal to $\frac{m'}{f}$, m' denoting the linear magnitude of the second image (301.). But the angle which the object subtends at the centre of the object-speculum (or to the naked eye, *q.p.*) is equal to that subtended by the first image at the same place, or equal to $\frac{m}{\text{F}}$; m denoting the linear magnitude of the first image. And the ratio of these angles, or the *magnifying power* of the telescope, is $\frac{m'}{m}\cdot\frac{\text{F}}{f}$. Or, since $\frac{m'}{m}=\frac{d'}{d}$ (73.),

$$\text{M} = \frac{\text{F}}{f}\cdot\frac{d'}{d}.$$

If we substitute in this expression for d and d' their values, $e - \text{F}$ and $e' - f$, and eliminate these quantities successively by means of the equation of the preceding article, we obtain the following expressions of the magnifying power:

$$\text{M} = \frac{\text{FF}'}{f(e - \text{F} - \text{F}')}, \quad \text{M} = \frac{\text{F}(e' - \text{F}' - f)}{\text{F}'f}.$$

The values of e and e', obtained from these expressions, are

$$e = \text{F} + \text{F}' + \frac{\text{FF}'}{f\text{M}}, \quad e' = \text{F}' + f + \frac{\text{F}'f\text{M}}{\text{F}},$$

values precisely analogous to those obtained (347.), and which determine the several intervals when the focal lengths and magnifying power are given.

The second of the preceding values of M furnishes a simple approximate value of the magnifying power. For, since the first image is nearly at the principal focus of the lesser speculum, and the second nearly at the vertex of the greater, $e' - F' - f = F$, *q.p.*; and this being substituted in the expression of M, there is

$$M = \frac{F^2}{F'f}.$$

The second image being inverted with respect to the first, and the first with respect to the object, it is evident that an object seen through this telescope will appear *erect*.

(383.) The quantity of light received by the object-speculum in the Gregorian telescope is to that received by the naked eye from the same object in the ratio of the difference of the areas of the two specula to that of the pupil of the eye; or in the ratio of $A^2 - A'^2$ to α^2, A and A' denoting the linear apertures of the two specula, and α that of the pupil. Wherefore, if m denote the ratio of the emergent to the incident light, after reflexion by the two specula and transmission through the eye-glass, the *quantity of light* received by the eye through the telescope will be

$$m\frac{A^2 - A'^2}{\alpha^2};$$

that received by the naked eye being unity. On this quantity depends the *penetrating power* of the instrument. It appears from what has been said (322.) that it cannot be increased indefinitely, but has a limit dependent upon the magnifying power.

The *apparent brightness* of an object seen through the telescope varies as the quantity of light in the image on the retina divided by the space over which it is diffused there. Now the area of the image on the retina is equal to M^2, that of the image on the retina of the naked eye being unity; wherefore the brightness of an object seen through the telescope is

$$m\frac{A^2 - A'^2}{\alpha^2 M^2};$$

the brightness of the same object to the naked eye being unity.

(384.) In order that the brightness of the centre of the field should be as great as possible, the lesser speculum must be of such dimensions as exactly to receive the whole cone of rays converging to the principal focus of the object-speculum. If it be less, it will not reflect the whole of the pencil reflected by the object-speculum : if it be greater, it will intercept more than is necessary of the parallel pencil incident upon the same. The aperture of the lesser speculum, therefore, is determined by the equation

$$A' = A\frac{d}{F} = A\left(\frac{e}{F} - 1\right).$$

The lesser speculum being of the magnitude here assigned, it is evident that the brightness of the field will diminish gradually from the centre. In order that the brightness may be uniform over the whole extent of the field, the magnitude of the lesser speculum must be somewhat greater than that above given, and its determination will offer no difficulty to the reader. But it is evident, from what has been said, that the brightness of the centre of the field will be thus diminished, and the loss of light on the whole will more than compensate the advantage gained.

The hole in the vertex of the object-speculum must not exceed the aperture of the lesser speculum; and it is evident that, as far as the principal pencil is concerned, no advantage will be gained by making it less. It is usual to make them equal, in order that the aperture of the eye-glass (which is limited by it), and therefore the field of view, may be as great as possible.

(385.) The extreme *field of view* in the Gregorian telescope is found by connecting the corresponding extremities of the lesser speculum and eye-glass by the line $B'b$ (see figure, page 367) : the intercepted portion of the image, fg, will determine the field. It will easily appear from similar triangles that $\frac{a - m'}{f} = \frac{m' - A'}{d'}$; in which A' denotes the semi-aperture of the lesser speculum, a that of the eye-glass, and m' half the linear magnitude of the second image. From this we have

$$x = \frac{ad - af}{d - f} = a - a\frac{f}{d},$$

nearly, since f may be neglected as inconsiderable in comparison with d. But if we substitute for a its value $a\frac{d}{F}$, obtained above, the value of x becomes

$$x = a - \frac{a}{m}.$$

But it appears from ([illegible]) that $\frac{x}{f} = m\frac{x}{F} = m\theta$, θ denoting half the field of view; therefore

$$\theta = \frac{x}{mf} = \frac{1}{mf}\left(a - \frac{a}{m}\right).$$

The second term of the quantity within the brackets is, in general, small in comparison with the first; and accordingly the field of view will be, nearly,

$$= \frac{a}{mf}.$$

([illegible].) Instead of a single eye-glass, it is usual to employ, in the Gregorian telescope, the Huyghenian double eye-piece, consisting of the eye-glass and field-glass (354.). The aperture of the latter being much greater than can be given to a single eye-glass, the field of view is much enlarged; while, at the same time, the errors of form and colour are considerably diminished. The theory of the telescope with this addition will be readily understood from what has been said in this section, combined with the results of the article already referred to.

A *field-bar*, or *diaphragm*, is usually introduced at the place of the anterior focus of the eye-glass, the diameter of which is equal to the linear magnitude of the image formed there. By this means all erratic light is prevented reaching the eye-glass. An *eye-stop* is also placed without the eye-glass, at the place of the intersection of the axes of the extreme pencils with the axis of the instrument; and the diameter of the eye-hole is to be taken equal to the linear magnitude of the image of the object-speculum, which will be formed at that point.

(387.) Some years after the invention of the Gregorian and Newtonian telescopes, and probably without any knowledge of what had been done in Britain, a Frenchman named Cassegrain published a description of the reflecting telescope which goes by his name. Cassegrain's telescope differs from the Gregorian merely in the form and position of the lesser speculum; this reflector being *convex* and placed *between* the object-speculum and its principal focus, at a distance from the latter a little less than its own focal length. In this construction, since the rays of each pencil are incident upon the lesser speculum converging to the points of an image behind it, and within the principal focus, they will, after reflexion, converge to the points of an image in front of the speculum. This image, being thrown near the vertex of the object-speculum, is viewed through an eye-glass placed at a distance from it equal to its own focal length; the object-speculum being perforated in the centre, as in the Gregorian telescope.

After what has been said of the Gregorian telescope, it will be needless to enter into detail on the theory of this construction. The results, as might easily have been anticipated, will differ from those obtained in the preceding articles merely in the sign of F', the focal length of the lesser speculum.

Objects seen through this instrument appear *inverted:* for the second image is erect with respect to the first, and therefore inverted with respect to the object. For this reason it is unsuitable to terrestrial purposes, unless with an erector eye-piece, an addition seldom made to the reflecting telescope.

The Cassegrainian construction has been little used. It appears, notwithstanding, to possess some very important advantages. It is shorter than a Gregorian of the same power by about double the focal length of the lesser speculum. Again, what is of still more importance, the spherical aberrations of the two specula lie in opposite directions, and therefore partially correct each other; whereas, in the Gregorian telescope, the aberration produced by the object-speculum is increased by that of the small reflector. It appears also from some experiments made by Captain Kater, that the Cassegrainian telescope surpasses the Gregorian of the same power and aperture in brightness as well as distinctness; a result which he attributes to the circumstance of the rays not having crossed one another at the focus. However this be, it has been ascertained by trial that a reflector of this form will sustain a higher magnifying power than a Gregorian with an object-metal of the same dimensions.

(388.) Doctor Smith's *reflecting microscope* is merely a Cassegrainian telescope adapted to near objects. It is evident that, if an object be brought very close to an instrument of this or of the Gregorian construction, the rays proceeding from it will be altogether intercepted by the second reflector before they reach the object-speculum. To obviate this, the aperture of the second speculum is increased, and the rays are admitted to the object-speculum through a circular hole in the centre. These rays are then reflected by the object-speculum, and converging to form an image a little within the focus of the convex speculum, they are a second time reflected, as in Cassegrain's telescope, and are made to converge to an image near the vertex of the object-speculum. Finally, the rays diverging from this second image pass through the aperture in the object-speculum, and are received by the eye-glass placed at a distance from it equal to its own focal length.

A small screen is placed between the apertures in the two reflectors, to intercept the rays which proceed directly from the object to the eye-glass.

APPENDIX.

I.

OF THE REFRACTION OF THE ATMOSPHERE.

(389.) We have seen (116.) that when a ray of light passes through any number of refracting media, bounded by parallel planes, its course will be a broken line, consisting of as many right lines as there are media inclined to the surfaces at different angles; and that the direction of the ray in the last medium, and therefore the total deviation, will be the same as if the light had been incident directly upon it out of the first. Now if the change of refractive power be *continuous*, the intervals of the successive parallel surfaces will be indefinitely small, and their number indefinitely great; consequently the polygonal line becomes a continuous curve, the direction of which, in any part of the varying medium, will be the same as if the ray had been incident directly upon it out of the original medium.

The atmosphere which surrounds the earth is a medium of this kind. For, since the density of the air decreases continually with the distance from the surface, the refractive power which is proportional to it must likewise decrease continually; and though the strata of equal density, and therefore of equal refractive power, are in fact spherical surfaces concentric with the earth, yet, on account of the small height of the sensible atmosphere compared with the radius of the earth, the deviation from parallelism throughout any space traversed by a ray of light (unless where the ray proceeds from a star near the horizon) must be inconsiderable. Hence, if we neglect the curvature of the atmospheric strata as inconsiderable throughout the space traversed by the ray, the direction of the ray in the last stratum, where it meets the earth, will be altogether independent of the law of variation of the density, and the same as if the light had been incident directly upon a homogeneous atmosphere of a density equal to that of the air at the surface.

(390.) The direction of the ray before its incidence upon the atmosphere marks the *real* position of the object; the direction of the ray when it meets the eye the *apparent* position; and the angle contained by these directions, or the total deviation of the ray, is the *refraction*. The law of its variation is readily determined on the principles above mentioned. For, if we suppose the ray to undergo refraction at the surface of a homogeneous atmosphere of the same density as that at the surface of the earth, the angle of refraction will be equal to the apparent zenith distance of the object, the surface of the atmosphere and that of the earth being parallel. Wherefore, if the apparent zenith distance be denoted by z, and the refraction or deviation by R, the angle of incidence will be $\text{z} + \text{R}$, and we have

$$\sin.(\text{z} + \text{R}) = m \sin.\text{z};$$

m denoting the ratio of the sines of incidence and refraction from a vacuum into air of the same density as that at the earth's surface. Or, expanding the first member of this equation, and making $\cos.\text{R} = 1$, $\sin.\text{R} = \text{R} \sin. 1''$, on account of the smallness of the angle R,

$$\sin.\text{z} + \text{R} \sin. 1'' \cos.\text{z} = m \sin.\text{z};$$

$$\therefore \text{R} = \frac{m - 1}{\sin. 1''} \tan.\text{z}.$$

The refraction varies therefore as the tangent of the apparent zenith distance.

The ray being bent towards the perpendicular to the surface, on entering a denser medium, it is evident that the apparent zenith distance is less than the true, or that the object appears elevated by the effect of refraction.

(391.) The refractive index of atmospheric air has been determined with great accuracy by the experiments of Biot and Arago. These authors found that, at the temperature of melting ice, and under a pressure measured by a column of mercury whose height was 76 *centimetres* (that is, at 29.93 inches barom. and 32° Fahrenheit), the value of $m - 1$ was .0002943. And dividing by sin. 1″, the coefficient of refraction was found to be 187.24 French seconds, or 60.66 English; so that

$$\frac{m-1}{\sin. 1''} = 60''.66.$$

The value of $m - 1$, however, varies with the density of the air, to which it is proportional; and consequently changes with the *temperature* and *pressure.* It appears from the experiments of Dalton and Gay-Lussac, that a column of air denoted by unity at the freezing point becomes 1.375 at the temperature of boiling water, the height of the barometer being the same; or that any volume of air increases by the fractional part .375, under an increase of temperature from 32° to 212°. Hence the increase of volume for each degree of temperature is .002083, the volume at the temperature 32° being unity. Wherefore, if x^0 denote the number of degrees of Fahrenheit above the freezing point, the volume at that temperature will be

$$1 + .002083x^0.$$

And, the density being inversely as the volume, we must divide the coefficient of refraction above given by this quantity, in order to correct for temperature.

Again, the temperature being given, the density of the air varies as the pressure, which is measured by the height of the mercury in the barometer. Consequently, the refraction being proportional to the density, the refraction at any pressure will be obtained by multiplying the refraction at a given pressure by the ratio of the heights of the mercury in the barometer. To correct for pressure, therefore, we must multiply the coefficient of refraction above given by the fraction

$$\frac{h}{29.93};$$

h denoting the height of the barometer in inches.

Applying, now, the corrections for temperature and pressure, the coefficient of refraction will be

$$\frac{m-1}{\sin. 1''} = \frac{h}{29.93} \times \frac{60''.66}{1 + .002083x^{\circ}}.$$

Where extreme accuracy is required, a correction for temperature is also applied to h, the height of the mercury in the barometer; the mercury expanding with the increase of heat. The correction, however, is extremely small.

It is a remarkable circumstance that the variations in the humidity of the atmosphere, which are considerable, have no sensible effect upon the refraction. The reason of this is that the density of aqueous vapour in a state of suspension is less than that of air, very nearly in the same proportion that its refractive power is greater. Consequently the effective refractive power of aqueous vapour is very nearly the same as that of atmospheric air; and therefore its admixture with the latter in varying proportions will not sensibly affect the quantity of refraction. In the corrections of refraction, therefore, it is unnecessary to attend to the indications of the hygrometer.

(392.) When the barometer stands at 29.6 inches, and Fahrenheit thermometer at 50°, the coefficient of refraction given in the preceding article becomes

$$\frac{29.60}{29.93} \times \frac{60''.66}{1.0375} = 57''.82;$$

and the general value of the coefficient, referred to this as a standard, will be

$$\frac{h}{29.6} \times \frac{1.0375 \times 57''.82}{1 + .002083x^{\circ}}.$$

Such is the numerical coefficient given by Dr. Brinkley *.

The coefficient of refraction derived from the French tables (barom. 29.6 inches, and Fahrenheit 50°) is 57''.72; and that obtained by Dr. Brinkley from upwards of 500 observations made at the observatory of Trinity College, Dublin, is 57''.56; so that the results derived from astronomical observation differ by but a very small quantity from those obtained by direct experiment. The author just mentioned considers the latter susceptible of a higher degree of precision than the former, and consequently more to be relied upon.

(393.) We have hitherto supposed the atmospherical strata to be parallel throughout the course of the ray. The results obtained on this supposition are sufficiently accurate for stars near the zenith; but when the zenith distance is considerable, there will be a correction required depending upon the sphericity of the strata. The deviation from parallelism being small, we may suppose, as before, the direction of the ray when it reaches the eye to be the same as if it had proceeded directly from a vacuum into air of the same density as that at the earth's surface. Wherefore, if θ denote the angle of refraction at the surface of the homogeneous atmosphere, supposed spherical and concentric with the earth, we have, as before (390.),

$$\text{R} = \frac{m-1}{\sin. 1''} \tan. \theta;$$

θ being no longer equal to z, the apparent zenith distance of the object. It is evident from the consideration of the triangle, formed by lines drawn from the centre of the earth to the extreme points of the refracted ray, that the sine of the angle of refraction is to the sine of the apparent zenith distance, as the radius of the earth to the radius of the exterior surface of the homogeneous atmosphere. Wherefore, if ρ denote the ratio which the height of the homogeneous atmosphere bears to the radius of the earth, there is

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* *Transactions of the Royal Irish Academy*, vol. xii.

$$\sin.\theta = \frac{\sin.z}{1+\varrho}.$$

This equation combined with the preceding will determine the refraction. In order to eliminate θ, we have

$$\cos.\theta = \sqrt{1 - \frac{\sin.^2 z}{(1+\varrho)^2}} = \frac{\sqrt{\cos.^2 z + 2\varrho}}{1+\varrho};$$

neglecting ϱ^2 in comparison with 2ϱ. Whence there is

$$\tan.\theta = \frac{\sin.z}{\sqrt{\cos.^2 z + 2\varrho}} = \frac{\tan.z}{\sqrt{1 + 2\varrho \sec.^2 z}}.$$

Or, expanding the denominator by the binomial theorem, and neglecting the powers of $\varrho \sec.^2 z$, after the first,

$$\tan.\theta = \tan.z\,(1 - \varrho \sec.^2 z);$$

$$\therefore \text{R} = \frac{m-1}{\sin.1''}\tan.z\,(1 - \varrho.\sec.^2 z).$$

The height of the homogeneous atmosphere = 5.095 miles at the temperature of 50° Fahr., and the radius of the earth = 3979 miles; whence the value of ϱ is found to be .00128. This quantity will have a correction depending upon the temperature; but the quantity itself being very small, its variation may be disregarded.

From the preceding it appears that the formula of refraction consists of two terms, the first of which is independent of the curvature of the earth, and the same as if the surface of the earth and the atmospheric strata were plane, and the second is the correction due to the sphericity. This latter term, at 80° zenith distance, amounts to about 14″ only, and at 40° zenith distance it is insensible.

(394.) The preceding formula for refraction has been obtained independently of any hypothesis respecting the law of variation of density of the atmosphere; and it is sufficiently accurate for all astronomical purposes as far as 74° zenith distance. For lower altitudes the law of variation of the density of the air will sensibly affect the quantity of atmospherical refraction; and as this law is as yet unknown, all formulæ for refraction hitherto given, beyond 74° zenith distance, must be considered as in some degree empirical. If the density of the air decreased in geometric progression (as would be the case on the supposition of a uniform temperature), Laplace has shown from theory that the refraction at the horizon would amount to 40′ nearly; the temperature being that of melting ice, and the height of the barometer 29.93 inches. If the density of the air diminished in arithmetical progression, the horizontal refraction would be about 30′. Now the observed horizontal refraction, which is about 35′ at the same temperature and pressure, is a mean between these results. Hence Laplace concludes that the law of the variation of the density of the air is nearly a mean between the abovementioned progressions; and, assuming a law of this nature, he has deduced a formula for the refraction which is applicable to all altitudes [*]. The French tables of refraction are constructed upon this formula.

At very low altitudes, however, the irregularity of refraction is so great, as to baffle all attempts to define it by a law. The variations in the state of the atmosphere at the surface of the earth, arising chiefly from sudden changes of temperature, produce variations in the horizontal refraction frequently exceeding its mean value Several instances have been recorded of unusually great refraction in the horizon. One of the most remarkable is that described in the *Philosophical Transactions* for the year 1798, in which an account is given of a refraction by which the French coast was rendered distinctly visible to the naked eye at the shore at Hastings, the distance being about fifty miles. By the aid of the telescope the French fishing-boats were plainly seen lying at anchor, and the buildings on the land distinctly discernible. Lalande mentions that the mountains of Corsica are occasionally visible from Genoa, a distance upwards of a hundred miles.

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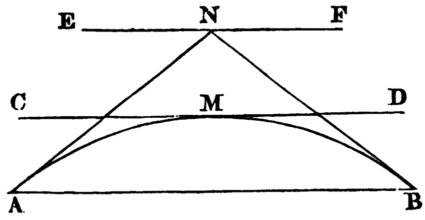
[*] See *Mecanique Celeste,* book 10, chap. 1. See also Dr. Brinkley's paper in the *Transactions of the Royal Irish Academy,* vol. xii.; in which the author has deduced, by a method as simple as the subject will admit, a formula which gives the refraction with sufficient accuracy as far as 80° zenith distance.

(395.) In a medium composed of parallel strata, whose refractive power is continually increasing or continually diminishing, a ray of light proceeding from the denser into the rarer part of the medium will be continually bent out of its course, and describe a curve which is concave towards the denser medium. Hence, as the angle of incidence with the strata of equal density is continually increasing, this angle may approach indefinitely to a right angle; and the ray, becoming parallel to these strata, may be bent back into the denser medium, and describe a second branch of the curve precisely similar to the first, the two branches of the curve intersecting the same surfaces at equal angles. The effect in such a case, therefore, will be the same as if the ray had suffered reflexion at some stratum of the rarer medium.

Thus let AMB be the curvilinear course of the ray, which is parallel to the strata of equal density at the point M: also let AB be one of these strata, intersecting the curve in A and B;



and let AN, BN, be tangents to the curve at these points. Then the ray will appear to be reflected from the stratum EF, which passes through N, the point of intersection of the tangents. For, since the strata of equal density intersect the two branches of the curve at equal angles, the angles NAB and NBA, and therefore their equals ANE and BNF, are equal. Hence, if the rays proceed from an object at B, and be refracted to the eye at A, the image of the object thus seen will be inverted. Also, the object being seen in the direction of the line AN, the tangent to the curve at the point where it meets the eye, it will appear elevated above its true position when the rarer medium is above the denser, and, on the contrary, depressed when it is below, the ray always inclining towards the denser medium.

It is evident, further, that if the eye be situated in the same stratum of the varying medium with the object, the rays proceeding directly from the object to the eye, through the medium of uniform density, may meet there with some of the curvilinear rays above mentioned proceeding out of the rarer medium; and that in this manner two images of the object will be visible, one *erect* by direct rays, and the other *inverted* by the rays which are as it were reflected from the rarer medium.

Thus, if into a rectangular vessel containing sulphuric acid water be gently poured, in such a manner that the two fluids may not mix, except throughout a small space in which a partial combination will take place, a medium will be produced between the acid and the water, decreasing gradually in density and therefore in refractive power as it approaches the water. If then any object be viewed through the upper part of the concentrated acid, it will be seen by the rays which proceed directly to the eye placed at an equal elevation on the opposite side; and the image thus formed will be erect. But it will be also seen by the rays which, proceeding from the object towards the rarer medium, are continually bent towards the denser, and finally intersect the direct rays at the place of the eye; and it is evident from what has been said that its image thus formed will be inverted. In this case, in which the upper medium is rarer than that below it, the second image is elevated with regard to the first: the contrary would have been the case had the rarer medium been the lower.

(396.) In the ordinary constitution of the atmosphere these phenomena are not exhibited. For, on account of the slow variation of density, the rays proceeding from the denser into the rarer medium would have an immense space to traverse in order to reach the eye situated in the same medium with the object: these rays, consequently, are extinguished in the atmosphere before they can reach the eye. When from any occasional cause, however, the change of density is very rapid, the appearances which we have been describing are plainly seen. It sometimes happens that the temperature of the sea is higher than that of the atmosphere, the former not cooling so rapidly as the latter, and therefore its temperature not being subject to the same sudden vicissitudes. When this is the case, the superficial stratum of air becomes warmer, and therefore also rarer, than that immediately above it; and the density will increase rapidly up to a certain height, and then diminish gradually. Now if the eye of a spectator be situated in the stratum of greatest density, and be directed to an object in the same stratum, he will observe two images: one erect, by the rays which are transmitted directly through the uniform medium; and the other inverted, by the rays which are, as it were, reflected upwards from the rarer medium. This second image will be depressed.

This phenomenon is frequently observed off the coast of Sicily, where it is attributed by the peasants to a supernatural agent, whom they call *Fata Morgana.* The French call it *mirage.* It is known in this country by the name of *looming;* and has been often seen at Leith, near Edinburgh, and on the north coast of Ireland, in the neighbourhood of the Giant's Causeway.

The same phenomenon, attended with some other remarkable appearances, is observed almost every day on the sandy plains of Lower Egypt; the dry soil of which becomes intensely heated by the midday sun, and thus rarefies the contiguous stratum of air much more than that above it. The plains at a distance become invisible, the light proceeding from objects situated there never reaching the eye. At the same time the villages on the heights, which are above the rarefied air, are seen directly through air of nearly uniform density, together with inverted images below them reflected from the rarer medium. Thus the plains themselves appear like a vast sea; above which the villages are seen like islands, together with their images beneath them reflected as it were from the surface of the water. As the traveller advances, the nearer parts of the plain become visible by the light transmitted thence directly to the eye; and thus the inundation seems to recede as if in mockery of his thirst *.

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* An account of this phenomenon was given by Monge, who accompanied Buonaparte in the Egyptian campaign. A mathematical explanation of its laws has been given by M. Biot in the Memoirs of the Institute for the year 1809.

Appearances similar to these may be produced artificially by heating a bar of metal, or a blackened stick, and looking along its edge at some distant object. The upper image, seen through a denser stratum of air, will be erect; while the lower, which is seen by light reflected from the rarer stratum, is inverted.

When the density of the strata decreases rapidly from the surface upwards, as will sometimes be the case when the temperature of the air over the sea is suddenly raised by the heat of the sun, the effects will be the opposite to those we have been describing, the inverted image being in this case *elevated*. Sometimes the appearances become more complicated, and *two extraordinary* images are seen, one of which is erect and the other inverted. Occasionally, when the object is distant, the ordinary image disappears, and one or two extraordinary images are seen suspended in the air. Appearances of this kind have been observed by Professor Vince at Ramsgate, and described in a paper read before the *Royal Society* in the year 1798. Others of the same nature have been described by Captain Scoresby, as witnessed by himself in the arctic seas. This author mentions an instance in which he recognized his father's vessel, when actually some leagues beyond the limit of direct vision, by means of an inverted image in the air, which was so well defined that he could distinguish every sail.

(397.) Before we leave the subject of atmospheric refractions, it will be necessary to say a few words on the theory of the *rainbow*, a phenomenon produced by the refraction of the sun's rays in the drops of falling rain. The first satisfactory account of the rainbow is that given by Antonius de Dominis, archbishop of Spalatro, in his work *De Radiis Visûs et Lucis*, published in the year 1611. This author shows in what manner the *interior bow* is formed by two refractions of the sun's light in the drops of rain, and one intermediate reflexion; and the *exterior bow* by two refractions, and two reflexions. This explanation, which was afterwards adopted by Descartes, was confirmed by experiments made with glass globes filled with water, and placed in such a manner as to exhibit the colours of the two bows. Neither of these authors, however, being acquainted with the true nature of colours, the full development of the theory of the rainbow was reserved for Newton.

(398.) When a pencil of solar rays is incident upon a spherical drop of water, and emergent after one reflexion within the drop, it is evident that the deviation of each ray of the pencil will depend upon its incidence; being equal to

$$\pi + 2\theta - 4\theta';$$

θ and θ' denoting the angles of incidence and refraction respectively (223.). Consequently the angle formed by the incident and emergent ray, which is equal to the excess of two right angles above the angle of deviation, will be

$$4\theta' - 2\theta.$$

Now this angle increases with the incidence, and then diminishes; and becomes a *maximum* when the angle of deviation is least, that is (224.) when

$$\cos. \theta = \sqrt{\frac{\mu^2 - 1}{3}}.$$

And since, from the nature of maxima and minima, the angle formed by the incident and emergent rays will then vary very little with the change of incidence, it is evident that the rays will emerge much more copiously at this inclination to the incident rays, than at any other; and consequently will affect the eye most strongly. Hence, to determine the angle which the emergent rays make with the incident, when refracted most copiously, we have only to find the value of θ, the angle of incidence, by the preceding formula; and from that to determine θ', the angle of refraction, by the equation

$$\sin. \theta = \mu \sin. \theta';$$

and finally, to substitute the values of θ and θ', thus obtained, in the formula $4\theta' - 2\theta$, the general expression of the angle required.

It is evident that the value of this angle will depend upon the index of refraction, μ; and therefore that it will be different for each of the different species of homogeneous light of which the solar light is composed. Consequently the rays of the different colours will emerge most copiously at different angles, and therefore become visible to the eye in different directions.

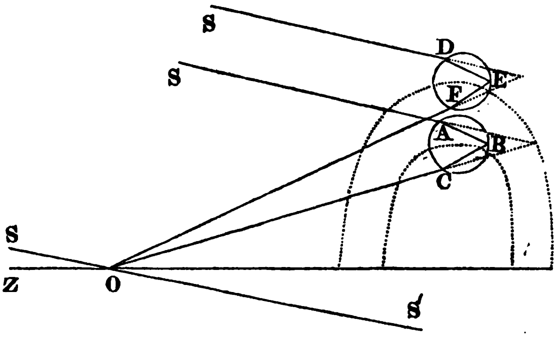
The values of the refractive index from air into water, for the red and violet rays respectively, are

$$\frac{108}{81} \text{ and } \frac{109}{81}.$$

And these values being substituted for μ in the preceding equations, by the aid of the trigonometrical tables we find the angles contained by the emergent red and violet rays with the incident, when most copiously refracted, to be respectively

$$42^\circ.2' \text{ and } 40^\circ.17'.$$

(399.) Now if O be the place of the eye of a spectator, SOS′ a line drawn through it in the direction of the sun's rays, and S′OF and S′OC equal to $42^\circ.2'$ and $40^\circ.17'$, respectively; the *red*



rays will be visible in the direction OF, the *violet* in the direction OC, and the rays of intermediate refrangibility in intermediate positions. For, the line SOS′ being parallel to the direction of the incident rays, the angles contained by the incident and emergent rays are equal to the angles S′OF, S′OC, contained by the emergent rays with this line; and therefore equal to $42^\circ.2'$ and $40^\circ.17'$ respectively. Wherefore the red rays will emerge most copiously in the direction FO, and the violet in the direction CO. But further, if the lines OF and OC revolve round the line OS′, which is called the *axis of vision*, it is evident that all the drops in the conical surface, generated by the revolution of the line OF, will transmit the *red* rays copiously to the eye; those in the conical surface, generated by the revolution of OC, the *violet;* and those in the intermediate surfaces the intermediate rays, in the order of their refrangibility. To the eye, therefore, there will appear a succession of coloured arches, in the order of the colours of the spectrum; the radii of the extremes of which, estimated by the angles which they subtend to the eye, are $42^\circ.2'$ and $40^\circ.17'$ respectively. This coloured band is called the *primary bow.*

(400.) The deviation of a ray of light which is incident upon a spherical drop of water, and emergent after *two reflexions* within the drop, is (223.)

$$2\pi + 2\theta - 6\theta';$$

and the angle contained by the incident and emergent rays, which is the excess of this above two right angles, is

$$\pi + 2\theta - 6\theta'.$$

Now this angle at first diminishes as the incidence increases, and afterwards increases; and becomes a *minimum* when the angle of deviation is least, that is (224.) when

$$\cos.\theta = \sqrt{\frac{\mu^2 - 1}{8}}.$$

And this equation, as is evident from what has been said above (398.) in the case of one reflexion, combined with the equation

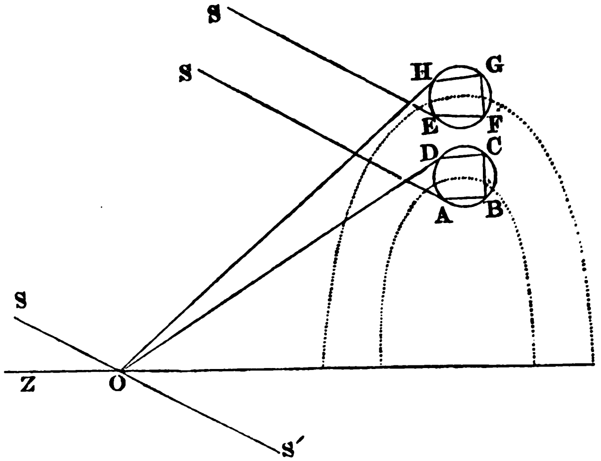
$$\sin.\theta = \mu \sin.\theta',$$

will determine θ and θ', the angles of incidence and refraction, when the rays emerge most copiously. These angles being known, the angle contained by the incident and emergent rays under the same circumstances, or $\pi + 2\theta - 6\theta'$, is determined. And it is evident that this angle will be different for each of the species of simple light; and therefore that the rays of the several colours will emerge most copiously at different angles.

To determine these angles for the *extreme red* and *violet* rays, we have only to substitute the corresponding values of μ, namely, $\frac{108}{81}$ and $\frac{109}{81}$, in the preceding equations; and by the aid of the trigonometrical tables we find the angles contained by the emergent red and violet rays with the incident, when refracted most copiously after two reflexions, to be

$$50^\circ.57' \text{ and } 54^\circ.7'.$$

(401.) Now O being the place of the spectator's eye, and SOS′ the line drawn through it in the direction of the incident rays, or the axis of vision, let the angles S′OD, S′OH, be taken equal to $50^\circ.57'$ and $54^\circ.7'$, respectively. Then, these angles being equal to the angles contained by the incident and emergent rays, it is evident that the *red* rays will emerge after two reflexions most copiously in the direction OD; and the *violet* in the direction OH.



Again, if the lines OD and OH be supposed to revolve round the axis of vision, os', they will generate two conical surfaces; the drops in the former of which will transmit the red rays most copiously; those in the latter the violet; and those in the intermediate surfaces the intermediate rays, in the order of their refrangibility. There will be seen therefore a succession of coloured arches, in the order of the colours of the spectrum, the radii of the extremes of which are $50^\circ.57'$ and $54^\circ.7'$, respectively. This is called the *secondary bow.*

The order of the colours in the secondary bow is the reverse of that in the primary; the red rays bounding the latter externally, and the former internally. For, in the interior or primary bow, the angle which the emergent rays form with the axis of vision is π — *angle of deviation;* in the secondary or exterior bow it is *angle of deviation* — π. Therefore the most refrangible rays, whose deviation is greatest, make the smallest angle with the axis of vision in the former case, and in the latter the greatest.

The secondary bow is much more faint than the primary, and is frequently invisible when the latter is distinctly seen. The reason of this is, that a portion of the light emerges from the drop at each reflexion, the reflexion being in no case total (223.); and consequently the pencil becomes weaker after each reflexion.

Sometimes a *tertiary* bow is seen, formed by two refractions and *three* reflexions; but, owing to the quantity of light lost at each reflexion, it is always extremely faint. The radii of its bounding arches are calculated from the formulæ of (223-4.), in the same manner as in the cases of the primary and secondary bows.

(402.) The *altitude* of any arc of the rainbow above the horizon is equal to the radius of that arc, diminished by the altitude of the sun; the two angles being referred to the same point of the sun's disc. But as the highest red arc of the primary bow is produced by rays flowing from the lowest point of the sun's disc, and the lowest violet arc by rays proceeding from the highest, the altitude of the *highest red* arc of the primary bow will be $42^\circ.2' - (A - 16')$, and that of the *lowest violet* arc $40^\circ.17' - (A + 16')$; A denoting the altitude of the sun's centre, and $16'$ being the apparent semi-diameter of the sun. And the *breadth of the bow*, which is the difference of these altitudes, is $1^\circ.45' + 32'$, or $2^\circ.17'$.

In like manner it appears that the altitude of the *highest violet* arc, in the secondary bow, is $54^\circ.7' - (\text{A} - 16')$, and that of the *lowest red* arc $50^\circ.57' - (\text{A} + 16')$. And the *breadth of the bow*, which is the difference of these angles, is $3^\circ.10' + 32'$, or $3^\circ.42'$.

The *primary* bow is invisible, when the altitude of the sun is equal to, or greater than, $42^\circ.18'$; and the *secondary*, when that altitude is equal to, or greater than, $54^\circ.23'$. When the sun is in the horizon, the altitude of either bow is greatest, and equal to the entire radius of the exterior arc [*].

(403.) In some cases a succession of coloured arches, of a reddish purple and green colour alternately, are seen within the primary rainbow, and immediately in contact with it. These are called *supernumerary rainbows*. The first of these is a reddish purple arch immediately in contact with the violet of the primary bow; next follows a green, then a purple, &c., diminishing in breadth and diameter like the coloured rings exhibited by thin plates.

Dr. Langwith has described a phenomenon of this kind, in which the supernumerary bows were continued as far as four alternations. The writer of these pages observed these arches, continued with great distinctness to as many successions, during the late autumn (1830). Similar arches, but much fainter, have been observed by M. Dicquemarre *without* the secondary bow. They have been explained by Dr. Young on the principle of the interference of the rays of light.

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[*] The preceding results rest upon the supposition that the view is limited by a plane drawn parallel to the horizon through the place of the eye. When this is not the case, or when there is as it were a *dip*, a greater part, or even the whole of the conical surfaces above mentioned, may transmit the rays of their respective colours to the eye. Such is the case when the phenomena of the bow are exhibited by the drops of dew in the grass. It is evident that the drops from which the rays are transmitted, in this case, are situated in the section of the cone formed by the surface of the ground; and that this section will be an ellipse, parabola, or hyperbola, according as the altitude of the sun is greater, equal to, or less than the semi-angle of the cone.

(404.) In the high northern latitudes the sun and moon are frequently seen surrounded by *halos*, or coloured circles, the diameters of which, measured by the angles which they subtend to the eye, are about 45° and 90° respectively. The colours exhibited by these circles are similar to those of the rainbow, but less distinct; and their external boundaries are very ill defined. The order of the colours is the same in both, the red being next the luminary. The same phenomenon is occasionally seen in our own latitude, especially in the colder months. These circles are sometimes accompanied by a horizontal circle passing through the sun; as also by two inverted and concentric arches of circles, touching the halos at their highest points, and whose common centre is, like that of the circle last mentioned, in the zenith. The latter arches are generally highly coloured: the horizontal circle is white. They are usually attended by brighter spots, or *parhelia*, at their points of intersection with the two circles first mentioned. Another bright spot, or *anthelion*, is sometimes seen at the point of the horizontal circle diametrically opposite to the sun. An interesting combination of these appearances is described by the astronomer Hevelius *, who observed it at Dantzic in the year 1661. In this beautiful phenomenon, which lasted for more than an hour, six mock suns were seen, some white and some coloured; and were attended with long and waving tails, which pointed from the true sun.

(405.) The phenomena of halos have been explained by Huyghens by the refraction of the globular particles of hail, a portion of the central parts of which he supposes to be formed of snow and opaque; and, by assigning certain magnitudes to these opaque nuclei, he has derived from theory the two circles above mentioned.

The most probable account of these phenomena, however, is that long since given by Mariotte, and of late revived by

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* *Mercurius in Sole visus.* Appendix, page 174.

Dr. Young. It is now well established that water has a natural tendency to crystallize at an angle of 60°; and that the common form of the crystal of ice is the triangular or the hexagonal prism of equal sides. Now Mariotte has calculated, on principles analogous to those by which the rainbow is explained, that prisms of ice suspended in the atmosphere, whose refracting angles are 60°, will produce a halo about the sun whose diameter is 45°.40′; which agrees very nearly with the dimensions obtained by the most accurate observations. The external halo may be explained, according to Dr. Young, by two successive refractions of the light through two prisms of the same refracting angle; by which means the deviation will be doubled, and a halo of 90° produced. Or, as Mr. Cavendish has suggested, this halo is more probably the effect of refraction through the rectangular terminations of the crystals; prisms of ice whose refracting angle is 90° producing a halo of 90°.28′, the refractive index of ice being 1.31. Or, lastly, this halo may be produced by refraction through the *faces* of four-sided equilateral prisms of ice; such prisms having been actually found.

The same explanation has been given also by M. Fraunhofer. According to this author, the diameters of the two principal halos, as derived from theory, are 45°.12′ and 90° respectively, the refractive index of ice being 1.32. These he calls *halos of the greater class.* The *coronæ,* or coloured rings, which are sometimes seen round the sun and moon, and are contiguous to one another and to the luminous body, are explained by the same author by the *inflexion* of the rays of light round the vesicles of vapour which float in the atmosphere; and he has calculated the diameter of these vesicles which will produce the rings observed. These rings are called by M. Fraunhofer *halos of the lesser class.*

The foregoing theory is well illustrated by an experiment adduced for that purpose by Dr. Brewster. If a few drops of a saturated solution of alum be spread upon a plate of glass, on evaporation it will crystallize, and a crust will be formed on the glass consisting of minute octohedral crystals. Now if the glass be held between the eye and the sun, or any other luminous object, three halos will be seen encircling

Cryolite	1.346
Vinegar	1.347
Port wine	1.351
Human blood	1.354
Albumen (extreme red)	1.360
White of hen's egg	1.360
Brandy and rum	1.360
Alcohol	1.372
Ether	1.374
Crystalline of human eye (mean index)	1.384
Muriatic acid (s. g. = 1.134)	1.392
Ditto (highly concentrated)	1.410
Nitric acid (s. g. = 1.48)	1.410
Sulphuric acid	1.435
Fluor spar	1.436
Alum (s. g. = 1.714)	1.458
Borax (s. g. = 1.714)	1.467
Spermaceti oil	1.470
Oil of olives	1.470
Naphtha	1.475
Oil of turpentine (s. g. = 0.885)	1.478
Carbonate of potash	1.482
Linseed oil (s. g. = 0.932)	1.482
Train oil	1.483
Obsidian	1.488
Oil of castor	1.490
Sulphate of iron (greatest refraction)	1.494
Honey	1.495
Rochelle salt	1.498
Camphor (s. g. = 0.996)	1.500
Stilbite	1.508
Sulphate of potash	1.509
Gum arabic	1.512
Needlestone from Faroe	1.515
Mesotype (least refraction)	1.516
———— (greatest)	1.522
Sulphate of zinc	1.517
Tartaric acid (least refraction)	1.518
Iceland spar (least refraction)	1.519

Iceland spar (greatest)	1.665
Myrrh	1.520
Glass of borax 1, silex 2	1.522
Leucite	1.527
Balsam of Gilead	1.529
Sulphate of copper (least refraction)	1.531
———————— (greatest)	1.552
Glass of borax	1.532
Manna	1.533
Caoutchouc	1.534
Plate glass	1.526 to 1.542
Arragonite (extraordinary index)	1.535
——— (ordinary)	1.693
Selenite (greatest refraction)	1.536
Felspar	1.536
Canada balsam	1.540
Carbonate of barytes (least refraction)	1.540
Apophyllite	1.543
Carbonate of strontia (least refraction)	1.543
————————— (greatest)	1.700
Dichroite	1.544
Petroleum	1.544
Rock salt (s. g. = 2.143)	1.545
Turpentine	1.545
Oil of tobacco	1.547
Mellite	1.547
Crown glass	1.531 to 1.563
Quartz (ordinary index)	1.548
——— (extraordinary)	1.558
Copal	1.549
Resin	1.553
Chalcedony	1.553
Amber (s. g. = 1.04)	1.556
Horn	1.565
Pink-coloured glass	1.570
Anhydrite (ordinary index)	1.577
———— (extraordinary)	1.622
Bottle-glass	1.582
Emerald	1.585

Pitch	1.586
Tortoise-shell	1.591
Beryl	1.598
Ruby-red glass	1.601
Meionite	1.606
Purple-coloured glass	1.608
Flint-glass	1.576 to 1.642
Colourless topaz	1.610
Green-coloured glass	1.615
Oil of cassia	1.632
Brazilian topaz (ordinary index)	1.632
— — (extraordinary)	1.640
Blue topaz	1.636
Yellow topaz	1.638
Euclase (ordinary index)	1.643
——— (extraordinary)	1.663
Sulphate of strontia	1.644
Sulphate of barytes (ordinary refraction)	1.646
Hyacinth-red glass	1.647
Red topaz	1.652
Mother pearl	1.653
Epidote (least refraction)	1.661
——— (greatest)	1.703
Tourmaline	1.668
Chrysolite (least refraction)	1.668
——— (greatest)	1.685
Orange-coloured glass	1.695
Boracite	1.701
Glass (lead 1, flint 2)	1.724
Deep red glass	1.729
Nitrate of silver (least refraction)	1.729
——— (greatest)	1.788
Glass (lead 3, flint 4)	1.732
Axinite	1.735
Nitrate of lead	1.758
Spinelle	1.758
Cinnamon stone	1.759
Chrysoberyl	1.760
Rubellite	1.773

Ruby	1.779
Jargon	1.782
Glass (lead 1, flint 1)	1.787
Pyrope	1.792
Sapphire (blue)	1.794
Labrador hornblende	1.800
Arsenic (extreme red)	1.811
Carbonate of lead (least refraction)	1.813
——— (greatest)	2.084
Sulphate of lead	1.925
Zircon (least refraction)	1.961
——— (greatest)	2.015
Calomel	1.970
Tungstate of lime (least refraction)	1.970
——— (greatest)	2.129
Glass (lead 2, flint 1)	1.987
—— (lead 3, flint 1)	2.028
Sulphur (native)	2.115
——— (melted)	2.148
Silicate of lead (extreme red)	2.123
Phosphorus	2.125 to 2.260
Glass of antimony	2.216
Plumbago (extreme red)	2.040 to 2.440
Nitrite of lead (ordinary refraction)	2.322
Octohedrite	2.500
Chromate of lead (least refraction)	2.500
——— (greatest)	2.950
Realgar	2.549
Read ore of silver	2.564
Diamond	2.439 to 2.755
Mercury (probable)	5.829

III.

TABLE OF DISPERSIVE POWERS *.

MEDIUM.	$\frac{\Delta\mu}{\mu - 1}$.	$\Delta\mu$.
Chromate of lead (greatest refraction) estimated	0.400	0.770
———————— exceeds	0.296	0.570
Chromate of lead (least refraction)	0.262	0.388
Realgar (melted)	0.267	0.394
——— (different kind)	0.255	0.374
Oil of cassia	0.139	0.089
Sulphur (after fusion)	0.130	0.149
Phosphorus	0.128	0.156
Carbonate of lead (greatest refraction)	0.091	0.091
——————— (least)	0.066	0.056
Green glass	0.061	0.037
Sulphate of lead	0.060	0.056
Deep red glass	0.060	0.044
Resin	0.057	0.032
Orange-coloured glass	0.053	0.042
Rock salt	0.053	0.029
Caoutchouc	0.052	0.028
Flint-glass	0.052	0.032
Deep purple glass	0.051	0.031
Flint-glass (another kind)	0.048	0.029
Chio turpentine	0.048	0.028
Flint-glass (another specimen)	0.048	0.028

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* This table is taken from that given by Dr. Brewster from the results of his own observations. The first column contains the names of the media; the third the values of $\Delta\mu$, or the difference of the refractive indices of the extreme red and violet rays; and the second the values of $\frac{\Delta\mu}{\mu - 1}$, or of the dispersive index.

Carbonate of strontia (greatest refraction)	0.046	0.032
Nitric acid	0.045	0.019
Tortoise shell	0.045	0.027
Horn	0.045	0.025
Canada balsam	0.045	0.024
Nitrous acid	0.044	0.018
Pink-coloured glass	0.044	0.025
Jargon (greatest refraction)	0.044	0.045
Muriatic acid	0.043	0.016
Gum copal	0.043	0.024
Burgundy pitch	0.043	0.024
Oil of turpentine	0.042	0.020
Felspar	0.042	0.022
Amber	0.041	0.023
Stilbite	0.041	0.021
Spinelle ruby	0.040	0.031
Calcareous spar (greatest refraction)	0.040	0.027
Bottle glass	0.040	0.023
Sulphate of iron	0.039	0.019
Diamond	0.038	0.056
Olive oil	0.038	0.018
White of an egg	0.037	0.013
Gum myrrh	0.037	0.020
Beryl	0.037	0.022
Obsidian	0.037	0.018
Ether	0.037	0.012
Selenite	0.037	0.020
Alum	0.036	0.017
Oil of castor	0.036	0.018
Sulphate of copper	0.036	0.019
Crown glass (very green)	0.036	0.020
Gum arabic	0.036	0.018
Water	0.035	0.012
Aqueous humour of haddock's eye	0.035	0.012
Vitreous humour of same	0.035	0.012
Rubellite	0.035	0.027
Leucite	0.035	0.018
Epidote	0.035	0.024
Garnet	0.033	0.027

Pyrope	0.033	0.026
Chrysolite	0.033	0.022
Crown glass	0.033	0.018
Plate glass	0.032	0.017
Sulphuric acid	0.031	0.014
Tartaric acid	0.030	0.016
Borax	0.030	0.014
Axinite	0.030	0.022
Alcohol	0.029	0.011
Sulphate of barytes	0.029	0.019
Tourmaline	0.028	0.019
Carbonate of strontia (least refraction)	0.027	0.015
Rock crystal	0.026	0.014
Emerald	0.026	0.015
Calcareous spar (least refraction)	0.026	0.016
Sapphire (blue)	0.026	0.021
Chrysoberyl	0.025	0.019
Topaz (bluish, Cairngorm).	0.025	0.016
——— (blue, Aberdeenshire)	0.024	0.025
Sulphate of strontia	0.024	0.015
Fluor spar	0.022	0.010
Cryolite	0.022	0.007

THE END.

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